Incentive Problem in Intergovernmental Transfers: Differences between Two Infinitely Iterated Leadership Models *

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Abstract

The authors deal with a certain type of timing problem in the allocation of central government subsidies to local governments, called the "soft budget constraint problem" in fiscal science. Many studies have claimed that central government incentives provided to bail out local governments cause distortion. This paper will show that this assertion is problematic by assuming that the central and local governments have an infinite-horizon view and by adopting the Markov perfect equilibrium (MPE) as a solution concept. The two models in this paper can be interpreted as dynamic social dilemmas. One is analogous to the tragedy of the commons, e.g., Levhari and Mirman (1980). The other shares some of the features of a dynamic social dilemma, and the two models are the same barring the timing of players' moves; despite this, the MPE behavior is quite different. By comparing these models, this paper will show that competitive restriction may improve social welfare.

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1 Introduction

In the paper, we deal with a certain type of timing problem in the allocation of subsidies by the central government to local governments. We will show that the path of occurrence of the distortion caused by subsidization varies according to the timing of the subsidy offer. Our timing problem is called "time inconsistency" in macroeconomics, which is addressed by Kydland and Prescott (1977). They conclude that policymakers fail to commit to a low inflation rate in advance since the level of unemployment is reduced by effecting high inflation. The same problem is called a "soft budget constraint (SBC) problem" in fiscal science. A considerable section of the literature on the SBC problem supports that central-government-sponsored bailouts for local governments through subsidies cause distortion. This paper will attempt a different interpretation of the SBC problem by assuming that central and local governments have an infinite-horizon view, which is standard in macroeconomics. Moreover, we will find that our models have the same structure as social dilemmas.

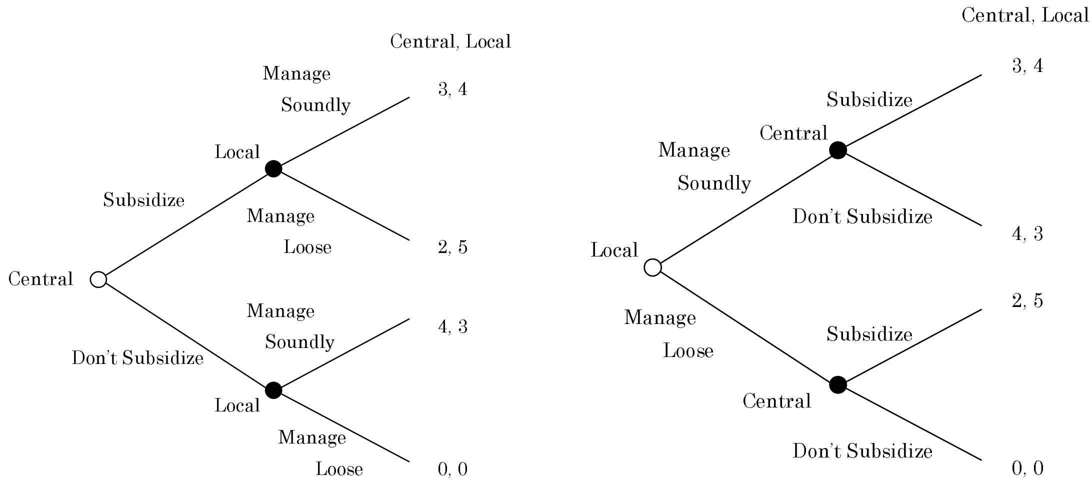
In fiscal science, it has been indicated that the interregional redistribution policy of the central government causes incentive problems such as excess expenditure or excess debt. These problems are referred to as SBCs, the origin of which is related to the analyses of distortions that resulted from bailouts of loss-making state-owned enterprises in a socialized economy (see Kornai (1979, 1980)).

SBC has been applied to the problem of subsidies provided to local governments by the central government and has been discussed as a cause of distortions resulting from the redistribution. For example, Wildasin (1997) and Akai and Sato (2005) attempt an analysis of a situation wherein the central government provides ex-post subsidy relief to areas in which consumption of public and private goods is in short supply compared with other areas. They conclude that under such conditions, the budget constraints of local governments "soften," and the supply of public goods and the rate of tax are distorted.

However, this result is not necessarily found only in a single-year model. Goodspeed (2002) can be considered to suggest this fact. If relations between the central and local governments are maintained over the years, present decision-making by local governments should have an influence on future decisions on central government subsidies. It can be also said that the present decision on the subsidy by the central government should have an influence on future decision-making of local governments. The point is that even if there is no distortion resulting from subsidy problems in the single period model, distortion can be caused by simply shifting those problems to a multiperiod model. Furthermore, compared to one-period model generating distortion, there could be different mechanisms generating distortion in some multiperiod models.

Taking this into account, we analyze problems of subsidy systems from central to local governments in "infinitely iterated leadership models." We use the term "infinitely iterated model" in the sense that this model is not simply a repeated game but has a state variable (vector in our model). Some of the literature refers to such models as "difference games," e.g., de Zeeuw and van der Ploeg (1991). While differential games are dynamic games in continuous time, difference games are dynamic games in discrete time. We use difference games to describe the timing of all the players' behaviors with fair accuracy. In this paper we suppose that there are two situations wherein local governments' decision-making on the supply of local public goods and on local bond issues, and the central government's decisions on the delivery of subsidies are made with different timings. In one situation, the central government decides the subsidy as the first move of every period, and the local governments then supply local public goods; this is called the "central leadership (CL)" model. In the other situation the order is inverted; this is called the "decentralized leadership (DL)" model.

The CL and DL models are used to explain the SBC problem between central and local governments. Toy examples of these are described in Figure 1. In the subgame perfect equilibrium of the CL model, the central government plays "don't subsidize" and the local government plays "manage soundly." By contrast, the local government plays "manage loose" and the central government plays "subsidize" in the equilibrium of the DL model. Therefore, the above examples show that a distortion occurs if the central government can, before choosing an action, observe the actions of the local government. This is the usual explanation of the



(a) One-shot CL model (b) One-shot DL model

Figure 1: Toy examples.

SBC problem in the context of intergovernmental transfer, hereinafter called "basic SBC distortion." On the other hand, in the above CL model no inefficiency is found.

However, inefficiency may occur in multiperiod CL models. In our model, each local government decides its local tax and local bonds, and the central government decides the subsidies to the local governments. In the model, an intertemporal distortion may occur, that is, the local government issues excess bonds to achieve a larger current consumption. In actuality, in two-period versions, we can show that an intertemporal distortion will occur even in the CL model since, in the first period, the local governments anticipate the second period's subsidies from the central government, which depend on the outstanding local bonds in the first period.[1] On the other hand, an intertemporal distortion and the basic SBC distortion will occur in the DL model. Moreover, we can show that the sum of the two types of distortion in a DL model is more aggravated than the intertemporal distortion in a CL model.

However, it is problematic that we admit to the above outcome in two-period

[1]We will show that an intertemporal distortion occurs in the infinitely iterated CL model. However, the reason for the distortion is quite different from that in the two-period CL model.

models because of the following reason: In general, an agent with a concave utility function equalizes its consumption for all periods in a multiperiod model. Thus, an increase or decrease in consumption in a single period due to intertemporal distortion may affect the consumption in all the other periods. If this effect, which occurs in all periods, accumulates and the intertemporal distortion in a CL model is more than that in a DL model, the welfare in the CL model may fall below that in the DL model.

For this reason, we should consider longer-period models. Consequently, we will deal mainly with infinite-period models in this paper; the reasons are as follows. First, infinite-period models have been used to analyze agent behavior in macroeconomics since Ramsey (1928). In particular, it seems plausible that a governments actions would be based on a long-term perspective. The second reason is to avoid the same effect as in any finitely repeated prisoner's dilemma, which is caused by the existence of a final period.

The Markov perfect equilibrium (MPE) is adopted as a solution concept in this paper. This concept is a refinement of the Nash equilibrium and has a high suitability to dynamic programming. Furthermore, Maskin and Tirole (2001) states that the MPE has many favorable properties, e.g., it requires only the coarsest of information. Any individually rational outcome is achieved by a trigger strategy as a subgame perfect equilibrium in an infinitely repeated game. On the other hand, no deviant player may be penalized in our setting as the Markov strategy only depends on the state variable in the adjacent period. Hence, it is impossible to detect any deviation if a state variable of any value is possible in period zero. Therefore, it is also impossible to penalize the deviant player without adopting a strategy that immediately penalizes in case the state variable attains a particular value even in period zero. We rationalize the use the MPE as follows: Although the folk theorem may predict Pareto-efficient outcomes, some of the literature on public finance suggest the inefficiency of local governments, e.g., Pettersson-Lidbom and Dahlberg (2005) and Doi and Ihori (2006). Doubtless, it is unknown which outcome is realized since the folk theorem also predicts a considerable number of other inefficient outcomes. Moreover, although every deviant player must be penalized by all the other players in any trigger strategy (or an analogous strategy such as a stick-and-carrot strategy) equilibrium, in a practical sense, no (probably local) government seems to be penalized by the other governments when it raises its local taxes and/or increases its public debt.

The models in this paper can be interpreted as dynamic social dilemmas. Levhari and Mirman (1980) and Dutta and Sundaram (1993) study dynamic common-property resource games. Velasco (2000) studies a model concerning fiscal policy that is essentially identical to that in the above literature. Though the infinitely iterated CL model is different from these models, it eventually has a solution that indicates the same behavior as that of these models. On the other hand, although the infinitely iterated DL model is the same as an infinitely iterated CL model barring the timing of the players' move and also shares some of the features of a social dilemma, its equilibrium behavior is different from that described the above literature.

Ortigueira (2006) has studied the relation between the optimal tax policy and the timing of actions. Ortigueira (2006) has the same purpose as our paper, in the sense that it attempts to study the importance of the timing of actions by comparing MPEs. Although Ortigueira (2006) has taken a numerical approach, our models will be solved analytically.

This paper is organized as follows: In Sections 2 and 3, we introduce infinitely iterated CL and DL models and calculate the MPEs. Section 4 is devoted to the description of finitely iterated models and attempts to justify the infinitely iterated models of the previous sections. In Section 5, we attempt a comparison between the infinitely iterated CL and DL models of the previous sections. Section 6 concludes this paper, and the last section includes some proofs.

2 Infinitely Iterated CL Model

The infinitely iterated CL model (or "CL model" for short) represents the following situation: In each year, the central government decides the subsidies to residents in each region before the local governments decide the outstanding local bonds and local taxes; and this process is repeated over many years.

2.1 Definition of the Game and the Equilibrium

Environment The economy contains two regions: region 1 and region 2. There are the central and two local governments. A typical region or a local government is represented by i. Each region consists of a representative resident with an infinite lifespan. The resident in region i earns one unit of income every period and his or her preference is represented by a utility function,

$$U^i(\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}}) = \sum_{t=1}^{\infty} \beta^{t-1} u^i(c_t^i, g_t^i),$$

where $\mathcal{T} = \{1, 2, \cdots\}$, c_t^i indicates private goods consumption, and g_t^i indicates the local public goods[2] supply in period t. They are assumed to be nonnegative. In each period, the players' decisions in the stage game are as follows:

(1) The central government decides subsidies (z_t^1, z_t^2) to each resident to satisfy $z_t^1 + z_t^2 = 0$. (First move in period t.)

(2) Both local governments simultaneously decide outstanding local bonds x_t^i and local taxes y_t^i. (Second move in period t.)

(3) Private goods consumption c_t^i and local public goods supply g_t^i are realized, where c_t^i and g_t^i satisfy the following equations:

$$c_t^i = 1 - y_t^i + z_t^i, \quad (1)$$

$$g_t^i = y_t^i + x_t^i - (1+r)x_{t-1}^i. \quad (2)$$

In the last equation, $r \in (0, 1)$ is an interest rate that is invariant with time and is set for all regions. Assume that the sum of the outstanding local bonds is constrained by the discounted sum of the future incomes in both regions ($\frac{2}{r}$); namely,

$$(\forall t \in \mathcal{T}) \ \ x_{t-1}^1 + x_{t-1}^2 \leq \frac{2}{r}, \quad (3)$$

where x_0^1 and x_0^2 are given for all the residents in the economy. In this economic environment, if $x_{\tau-1}^1 + x_{\tau-1}^2 = \frac{2}{r}$ for some $\tau \in \mathcal{T}$, $x_t^1 + x_t^2 = \frac{2}{r}$ for all $t \geq \tau$ because of the budget constraint,

[2]Strictly speaking, the goods are not public goods. We could give g_t^i the properties of public goods; however, this would add unnecessary complexity to our assertion.

$$(\forall t \in \mathcal{T}) \;\; 2 + x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2) = c_t^1 + c_t^2 + g_t^1 + g_t^2 \geq 0.$$

Furthermore, the sum of local taxes must be smaller than or equal to 2, namely,

$$(\forall t \in \mathcal{T}) \;\; y_t^1 + y_t^2 \leq 2, \tag{4}$$

since $z_t^1 + z_t^2 = 0$ and $c_t^i = 1 - y_t^i + z_t^i \geq 0$ for $i = 1, 2$.

The central government aims to make the social welfare, $\sum_{i=1}^2 \theta^i U^i(\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}})$, as large as possible, where $\theta^1 \geq 0, \theta^2 \geq 0$; while local government i aims to maximize $U^i(\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}})$.

History Let $\mathcal{B} \subset (\mathbb{R}^{\mathcal{T}})^2$, $\mathcal{Q} \subset (\mathbb{R}^{\mathcal{T}})^2$ and $\mathcal{S} \subset (\mathbb{R}^2)^{\mathcal{T}}$ denote a set of the history of admissible outstanding local bonds, that of local taxes, and that of subsidies from the central to local governments, respectively.

For a history of subsidies $\mathbf{z} = ((z_1^1, z_1^2), (z_2^1, z_2^2), \cdots) \in \mathcal{S}$, let $\mathbf{z}_t$ denote a history of subsidies from the first to period t, that is, $((z_1^1, z_1^2), (z_2^1, z_2^2), \cdots, (z_t^1, z_t^2))$. Similarly, for a history of local taxes (local bonds) $\mathbf{y} \in \mathcal{Q}$ ($\mathbf{x} \in \mathcal{B}$), $\mathbf{y}_t$ ($\mathbf{x}_{t-1}$) denotes the first t elements of the history. For all $t \in \mathcal{T}$, sets of all these elements are denoted by $\mathcal{S}_t$, $\mathcal{Q}_t$, and $\mathcal{B}_{t-1}$. Note that $\mathcal{B}_t$ is a subset of $(\mathbb{R}^{t+1})^2$, while $\mathcal{Q}_t$ and $\mathcal{S}_t$ are subsets of $(\mathbb{R}^t)^2$ and $(\mathbb{R}^2)^t$, respectively.[3]

We define the sets of histories ($\mathcal{H}$, $\mathcal{H}_t$, and $\mathcal{H}_t^C$) as follows.

$$\begin{aligned}
\mathcal{H} &\equiv \mathcal{B} \times \mathcal{Q} \times \mathcal{S}, \\
\mathcal{H}_t &\equiv \begin{cases} \mathcal{B}_0 & \text{if } t = 0 \\ \mathcal{B}_t \times \mathcal{Q}_t \times \mathcal{S}_t & \text{if } t \in \mathcal{T} \end{cases}, \\
\mathcal{H}_t^C &\equiv \begin{cases} \mathcal{B}_0 \times \mathcal{S}_1 & \text{if } t = 1 \\ \mathcal{B}_{t-1} \times \mathcal{Q}_{t-1} \times \mathcal{S}_t & \text{if } t \in \mathcal{T}/\{1\}. \end{cases}
\end{aligned}$$

$\mathcal{H}_t$ denotes the set of histories up to the second move in period t, and $\mathcal{H}_t^C$, the set of histories up to the first move in period t.

Strategy Set The local governments' strategy sets $\mathcal{A}^{C1}$ and $\mathcal{A}^{C2}$ and the central government's strategy set $\mathcal{A}^{C0}$ are defined as follows. For $i = 1, 2$,

[3]The formal definitions of $\mathcal{B}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{B}_{t-1}$, $\mathcal{Q}_t$, and $\mathcal{S}_t$ may be found in Appendix A.

$$\mathcal{A}^{Ci} \equiv \Big\{((b_1,q_1),(b_2,q_2),\cdots) \in \Pi_{t\in\mathcal{T}}\left(\mathbb{R}^{\mathcal{H}_t^C}\right)^2 \Big| (\forall t \in \mathcal{T})(\forall h \in \mathcal{H}_t^C)$$
$$1 - q_t(h) + z_t^i \geq 0,\ q_t(h) + b_t(h) - (1+r)x_{t-1}^i \geq 0 \textit{ where}$$
$$h = ((x_0^1,\cdots,x_{t-1}^1),(x_0^2,\cdots,x_{t-1}^2),\mathbf{y}_{t-1},((z_1^1,z_1^2),\cdots,(z_t^1,z_t^2)))\Big\},$$
$$\mathcal{A}^{C0} \equiv \Big\{((s_1^1,s_1^2),(s_2^1,s_2^2),\cdots) \in \Pi_{t\in\mathcal{T}}(\mathbb{R}^{\mathcal{H}_{t-1}})^2 \Big| (\forall t \in \mathcal{T})(\forall h \in \mathcal{H}_{t-1})$$
$$s_t^1(h) + s_t^2(h) = 0\Big\}.$$

For given $((s_1^1,s_1^2),(s_2^1,s_2^2),\cdots) \in \mathcal{A}^{C0}$, $s_t^i \in \mathbb{R}^{\mathcal{H}_{t-1}}$ represents a function that assigns subsidies to the resident in region i in period t. In period t, the central government decides subsidies depending on the history up to period $(t-1)$. On the other hand, for given $((b_1,q_1),(b_2,q_2),\cdots) \in \mathcal{A}^{Ci}$, $b_t \in \mathbb{R}^{\mathcal{H}_t^\ell}$ ($q_t \in \mathbb{R}^{\mathcal{H}_t^\ell}$) represents the outstanding local bonds (local taxes) in period t. In period t, the local governments decide the outstanding local bonds and local taxes depending on the history up to period $(t-1)$ and on the central government's decision in period t.

However, for some $t \in \mathcal{T}$, $((b_1^1,q_1^1),(b_2^1,q_2^1),\cdots) \in \mathcal{A}^{C1}$, $((b_1^2,q_1^2),(b_2^2,q_2^2),\cdots) \in \mathcal{A}^{C2}$, and $h \in \mathcal{H}_t^C$, it may be that $b_t^1(h) + b_t^2(h) > \frac{2}{r}$. Therefore, to satisfy (3), if $b_t^1(h) + b_t^2(h) > \frac{2}{r}$, the outstanding local bonds (x_t^i) and local taxes (y_t^i) of local government i in period t are assumed to be[4]

$$x_t^i = -1 - z_t^i + \frac{1+r}{2}\left(\frac{2}{r} + x_{t-1}^i - x_{t-1}^j\right), \tag{5}$$
$$y_t^i = 1 + z_t^i - \frac{1+r}{4}\left(\frac{2}{r} - x_{\tau-1}^1 - x_{\tau-1}^2\right). \tag{6}$$

Definition of Equilibria Let $\mathcal{A}^C$ denote $\mathcal{A}^{C0} \times \mathcal{A}^{C1} \times \mathcal{A}^{C2}$. When a combination of strategies, $\mathbf{a} = (a^0,a^1,a^2) \in \mathcal{A}^C$, and a history up to period τ, $h \in \mathcal{H}_\tau$, are given, the sequence of consumption in region i after period $(\tau+1)$, $\mathbf{w}^i(\mathbf{a},h) = (\{c_t^i\}_{t\in\mathcal{T}},\{g_t^i\}_{t\in\mathcal{T}})$, can be determined uniquely for $i = 1,2$. For example, when $a^0 = ((s_1^1,s_1^2),(s_2^1,s_2^2),\cdots)$, $a^i = ((b_1^i,q_1^i),(b_2^i,q_2^i),\cdots)$, $h = ((\mathbf{x}_\tau^1,\mathbf{x}_\tau^2),(\mathbf{y}_\tau^1,\mathbf{y}_\tau^2),\mathbf{z}_\tau)$, and $\mathbf{x}_\tau^1 = (x_0^i,\cdots,x_\tau^i)$, c_t^i and g_t^i are determined by (1) and (2), where $x_0^i = x_\tau^i$;

[4] Under this determination of x_t^i and y_t^i, $x_t^1 + x_t^2 = \frac{2}{r}$, $y_t^i + x_t^i - (1+r)x_{t-1}^i \geq 0$, and $1 - y_t^i + z_t^i \geq 0$. Moreover, $c_T^i = g_T^i = 0$ for $T \geq t+1$ and for $i = 1,2$. The equilibrium strategy presented in Proposition 2 necessarily satisfies $b_t^1(h) + b_t^2(h) \leq \frac{2}{r}$. The result does not depend on the method used to determine x_t^i and y_t^i when $b_t^1(h) + b_t^2(h) > \frac{2}{r}$, as long as they satisfy $x_t^1 + x_t^2 = \frac{2}{r}$, $y_t^i + x_t^i - (1+r)x_{t-1}^i \geq 0$, and $1 - y_t^i + z_t^i \geq 0$.

and for $t \in \mathcal{T}$ at which $b^1_{\tau+t} + b^2_{\tau+t} \leq \frac{2}{r}$, x^i_t, y^i_t, and z^i_t are defined in the following manner:[5]

$$x^i_t = b^i_{\tau+t}(h^\ell_t), \quad y^i_t = q^i_{\tau+t}(h^\ell_t), \quad z^i_t = s^i_{\tau+t}(h_{t-1}), \quad \mathbf{z}_{\tau+t} = (\mathbf{z}^i_{\tau+t-1}, (z^1_t, z^2_t)),$$

$$h_0 = h, \quad h^\ell_t = ((\,\mathbf{x}^1_{\tau+t-1}, \mathbf{x}^2_{\tau+t-1}), \mathbf{y}^1_{\tau+t-1}, \mathbf{y}^2_{\tau+t-1}, \mathbf{z}_{\tau+t}),$$

$$\mathbf{x}^i_{\tau+t} = (\mathbf{x}^i_{\tau+t-1}, x^i_t), \quad \mathbf{y}^i_{\tau+t} = (\mathbf{y}^i_{\tau+t-1}, y^i_t), \quad h_t = (\mathbf{x}^1_{\tau+t}, \mathbf{x}^2_{\tau+t}, \mathbf{y}^1_{\tau+t}, \mathbf{y}^2_{\tau+t}, \mathbf{z}_{\tau+t}).$$

In a similar manner, when a combination of strategies and the history up to the first move in period τ, $h \in \mathcal{H}^C_t$, are given, the sequence of consumption after the period τ, $\mathbf{w}^i(\mathbf{a}, h)$, can be uniquely determined.

Definition 1 A combination of strategies $(a^{0*}, a^{1*}, a^{2*}) \in \mathcal{A}^C$ is a subgame perfect equilibrium of the CL model if it satisfies the following conditions:

$$(\forall t \in \mathcal{T})\,(\forall h \in \mathcal{H}_{t-1}) \;\; a^{0*} \in \arg\max_{a \in \mathcal{A}^{C0}} \sum_{i=1,2} \theta^i U^i(\mathbf{w}^i((a, a^{1*}, a^{2*}), h)),$$
$$(\forall t \in \mathcal{T})\,(\forall h \in \mathcal{H}^C_t) \;\; a^{1*} \in \arg\max_{a \in \mathcal{A}^{C1}} U^1(\mathbf{w}^1((a^{0*}, a, a^{2*}), h)),$$
$$(\forall t \in \mathcal{T})\,(\forall h \in \mathcal{H}^C_t) \;\; a^{2*} \in \arg\max_{a \in \mathcal{A}^{C2}} U^2(\mathbf{w}^2((a^{0*}, a^{1*}, a), h)).$$

Furthermore, the MPE is defined. When functions $f \in \mathbb{R}^{\mathcal{H}_0}$ and $e \in \mathbb{R}^{\mathcal{H}^C_1}$ are given, the functions $\mathbf{f} = (f_1, f_2, \cdots) \in \Pi_{t\in\mathcal{T}} \mathbb{R}^{\mathcal{H}_{t-1}}$ and $\mathbf{e} = (e_1, e_2, \cdots) \in \Pi_{t\in\mathcal{T}} \mathbb{R}^{\mathcal{H}^C_t}$ can be uniquely determined in the following manner:

$$f_1 = f \quad \text{and} \quad e_1 = e \tag{7}$$

$$(\forall t \in \mathcal{T}) \;\; f_{t+1}(\mathbf{x}^1_t, \mathbf{x}^2_t, \mathbf{y}^1_t, \mathbf{y}^2_t, \mathbf{z}_t) = f(x^1_t, x^2_t), \tag{8}$$

$$(\forall t \in \mathcal{T}) \;\; e_{t+1}(\mathbf{x}^1_t, \mathbf{x}^2_t, \mathbf{y}^1_t, \mathbf{y}^2_t, \mathbf{z}_{t+1}) = e(x^1_t, x^2_t, z^1_{t+1}, z^2_{t+1}), \tag{9}$$

where $(\mathbf{x}^1_t, \mathbf{x}^2_t) \in \mathcal{B}_t$, $\mathbf{x}^i_t = (x^i_0, x^i_1, \cdots, x^i_t)$, $\mathbf{y}^i_t \in \mathcal{Q}_t$, and $\mathbf{z}_t = ((z^1_1, z^2_1), \cdots, (z^1_t, z^2_t)) \in \mathcal{S}_t$. Using (7) and (8), one can construct a strategy $a \in \mathcal{A}^{C0}$ from a combination of functions $(s^1, s^2) \in (\mathbb{R}^{\mathcal{H}_0})^2$ if $s^1 + s^2 = 0$. Hence, we state that a combination of functions $(s^1, s^2) \in (\mathbb{R}^{\mathcal{H}_0})^2$ satisfying $s^1 + s^2 = 0$ is a Markov strategy of the central government.

Similarly, for $i = 1, 2$, one can construct a strategy $a \in \mathcal{A}^{Ci}$ from a combination of functions $(b, q) \in (\mathbb{R}^{\mathcal{H}^C_1})^2$ if $1 - q(h) + z^i \geq 0$ and $q(h) + b(h) - (1+r)x^i \geq 0$, where

[5] If $b^1_{\tau+t} + b^2_{\tau+t} > \frac{2}{r}$, x^i_t and y^i_t are determined through (5) and (6).

$h = (x^1, x^2, z^1, z^2)$, by using (7) and (9). Hence, we state that a combination of functions $(b, q) \in (\mathbb{R}^{\mathcal{H}_0})^2$ satisfying $1 - q(h) + z^i \geq 0$ and $q(h) + b(h) - (1+r)x^i \geq 0$ is a Markov strategy of local government i.

Definition 2 A combination of Markov strategies $((s^1, s^2), (b^1, q^1), (b^2, q^2))$ is an MPE of the CL model if $(a^0, a^1, a^2) \in \mathcal{A}^C$ is a subgame perfect equilibrium, where a^0 is constructed from (s^1, s^2) by using (7) and (8) and, for $i = 1, 2$, a^i is constructed from (b^i, q^i) by using (7) and (9).

Note that the MPE is subgame perfect since Markov restriction is not imposed on the definition of strategy sets. As discussed in Blanchard and Fischer (1989, Chap.11), a subgame perfect equilibrium strategy of the central government is time consistent in the sense discussed by Kydland and Prescott (1977).

In the following, we make the following assumptions.

Assumption (i) $\beta(1 + r) < 1$; (ii) $u^i(c, g) = \ln c + \ln g$ for $i = 1, 2$; and (iii) $\theta^1 = \theta^2 = 1$.

2.2 Planning Problem

The problem of maximizing the social welfare is solved in this subsection. This problem is described in the following manner:

$$\max \sum_{t=1}^{\infty} \beta^{t-1} \sum_{i=1}^{2} \{\ln(1 - y_t^i + z_t^i) + \ln[y_t^i + x_t^i - (1+r)x_{t-1}^i]\}$$
$$\text{s.t} \quad 1 - y_t^i + z_t^i \geq 0, \ y_t^i + x_t^i - (1+r)x_{t-1}^i \geq 0,$$
$$x_t^1 + x_t^2 \leq \frac{2}{r}, \ z_t^1 + z_t^2 = 0, \ x_0^1 \text{ and } x_0^2 \text{ are given.}$$

Let us define the function $I \in (R_+)^{\mathcal{B}^0}$ as $I(x^1, x^2) = (1 + r)\left(\frac{2}{r} - x^1 - x^2\right)$. From the theory of dynamic programming,[6] we can solve the problem and obtain the following result.

Proposition 1 There exist x_t^i, y_t^i, z_t^i that solve the planning problem. The gross outstanding local bonds $x_t^1 + x_t^2$, private consumption $c_t^i = 1 - y_t^i + z_t^i$, and public goods supply $g_t^i = y_t^i + x_t^i - (1+r)x_{t-1}^i$ corresponding to all the solutions are the same. Furthermore, they satisfy the following equations:

[6] See Stokey, Lucas, and Prescott (1989) as a standard textbook.

$$I(x_t^1,\, x_t^2) = \beta(1+r)I(x_{t-1}^1,\, x_{t-1}^2), \tag{10}$$

$$y_t^1 + y_t^2 = 2 - \frac{1-\beta}{2} I(x_t^1,\, x_t^2), \tag{11}$$

$$c_t^1 = c_t^2 = g_t^1 = g_t^2 = \frac{1-\beta}{4} I(x_t^1,\, x_t^2),$$

$$U^1(\{c_t^1, g_t^1\}_{t\in\mathcal{T}}) = U^2(\{c_t^2, g_t^2\}_{t\in\mathcal{T}}) = V^*(x_0^1, x_0^2),$$

$$V^*(x_0^1, x_0^2) = \frac{2}{1-\beta} \ln\left(\frac{2}{r} - x_0^1 - x_0^2\right) + \delta^*,$$

$$\delta^* = \frac{2}{(1-\beta)^2}\,\{\beta\ln\beta + (1-\beta)\ln(1-\beta) + \ln(1+r) - 2(1-\beta)\ln 2\}.$$

$I(x^1, x^2)$ represents the net present value of the lifetime income in both regions (or the *gross lifetime income* for short), namely, the sum of today's gross income and the discounted future gross income $(2+\frac{2}{r})$ minus bond redemption$((1+r)(x^1+x^2))$. Since $\beta(1+r) < 1$, $x_t^1+x_t^2 \to \frac{2}{r}$ and $I(x_t^1, x_t^2) \to 0$ as $t \to \infty$ in the optimal solution.

When the gross lifetime income increases, the current consumption also increases. For this purpose, the net amount of bond issuance increases and, at the same time, the local tax decreases for an intratemporal balance. By a transformation of equation (10), $x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2) = (1-\beta)I(x_{t-1}^1, x_{t-1}^2) - 2$ follows. This equation along with equation (16) implies the abovementioned characteristics.

2.3 Markov Perfect Equilibrium

Proposition 2 For an arbitrary function $\hat{s} \in \mathbb{R}^{\mathcal{H}_0}$, $((\hat{s}^1, \hat{s}^2), (\hat{b}^1, \hat{q}^1), (\hat{b}^2, \hat{q}^2))$ is an MPE of the CL model, where for $(x^1, x^2, z^1, z^2) \in \mathcal{H}_1^C$,

$$\hat{s}^1 = -\hat{s}^2 = \hat{s}, \tag{12}$$

$$(\forall i = 1, 2) \quad \hat{q}^i(x^1, x^2, z^1, z^2) = 1 + z^i - \frac{1-\beta}{2(2-\beta)}\, I(x^1, x^2), \tag{13}$$

$$(\forall i = 1, 2) \;\; \hat{b}^i(x^1, x^2, z^1, z^2) = -(1 + z^i) + (1+r)x^i + \frac{1-\beta}{2-\beta}\, I(x^1, x^2). \tag{14}$$

Functional equations for the MPE, corresponding to the Bellman equation of a dynamic programming are as follows for $i = 1,\, 2$:

$$V^i(x^1, x^2) = F^i(x^1, x^2, s^1(x^1, x^2), s^2(x^1, x^2)),$$

$$F^i(h) = \ln(1 - q^i(h) + z^i) + \ln(q^i(h) + b^i(h) - (1+r)x^i) + \beta V^i(b^1(h), b^2(h)),$$
$$\sum_{i=1}^{2} V^i(x^1, x^2) = \max_{(z^1,z^2)\in\mathcal{S}_1} \sum_{i=1}^{2} F^i(x^1, x^2, z^1, z^2),$$
$$F^1(h) = \max_{x,y}\{\ln(1 - y + z^1) + \ln(y + x - (1+r)x^1) + \beta V^1(x, b^2(h))\},$$
$$F^2(h) = \max_{x,y}\{\ln(1 - y + z^2) + \ln(y + x - (1+r)x^2) + \beta V^2(b^1(h), x)\},$$

where $h = (x^1, x^2, z^1, z^2) \in \mathcal{H}_1^C$ and $((s^1, s^2), (b^1, q^1), (b^2, q^2))$ is a combination of Markov strategies. The functional equations are satisfied by $V^1 = V^2 = V^C$ (V^C will be presented in Corollary 1) and $((\hat{s}^1, \hat{s}^2), (\hat{b}^1, \hat{q}^1), (\hat{b}^2, \hat{q}^2))$ in Proposition 2. However, a proof for the Proposition is necessary, since we cannot directly apply the theory of dynamic programming, e.g.,Theorem 4.2 of Stokey et al. (1989). See Appendix B for the proof.

The local governments' strategies are not controllable by the central government's strategies in the following sense: The central government can choose any Markov strategy. Moreover, for an arbitrary strategy $a \in \mathcal{A}^{C0}$ of the central government, $(a, \hat{a}^1, \hat{a}^2)$ is a subgame perfect equilibrium, where $\hat{a}^1$ and $\hat{a}^2$ are constructed from $(\hat{b}^1, \hat{q}^1)$ and $(\hat{b}^2, \hat{q}^2)$, respectively, by using (7) and (9). Which is to say that no matter how the central government varies its strategies, each local government has no incentive to deviate as long as the strategies of both local governments satisfy (13) and (14) since their strategies constitute an MPE. In our model, each local government has no interest in the delivery rule of the subsidies but is only concerned with the amount of the subsidies. Moreover, each local government regards these subsidies as an increase in the resident's income.

Every local tax is dependent on the sum of the current income and subsidy as well as on the gross lifetime income. Every outstanding local bond is dependent on previous bonds, the sum of the current income and subsidy, and the gross lifetime income. The local government decreases the outstanding local bonds by an amount equal to the increase in the subsidy; at the same time, the local government also increases the local tax by the same amount. On the other hand, as the gross lifetime income increases, the local government increases the net amount of bond issuance and simultaneously decreases the local tax.

For an arbitrary function $\hat{s} \in \mathbb{R}^{\mathcal{H}_0}$, every pair of Markov strategies satisfying (12), (13), and (14) constitutes an equilibrium. Therefore, Proposition 2 states there is a continuum of MPEs. However, the consumption path is the same for all the MPEs.

Corollary 1 The gross outstanding local bonds, private consumption, and public goods supply corresponding to the x_t^i, y_t^i, z_t^i determined by all the strategies in Proposition 2 are the same. Furthermore, they satisfy the following equations:

$$I(x_t^1, x_t^2) = \frac{\beta(1+r)}{2-\beta} I(x_{t-1}^1, x_{t-1}^2), \tag{15}$$

$$y_t^1 + y_t^2 = 2 - \frac{1-\beta}{2-\beta} I(x_{t-1}^1, x_{t-1}^2), \tag{16}$$

$$c_t^1 = c_t^2 = g_t^1 = g_t^2 = \frac{1-\beta}{2(2-\beta)} I(x_{t-1}^1, x_{t-1}^2),$$

$$U^1(\{c_t^1, g_t^1\}_{t\in\mathcal{T}}) = U^2(\{c_t^2, g_t^2\}_{t\in\mathcal{T}}) = V^C(x_0^1, x_0^2),$$

$$V^C(x_0^1, x_0^2) = \frac{2}{1-\beta} \ln\left(\frac{2}{r} - x_0^1 - x_0^2\right) + \delta^C,$$

$$\delta^C = \frac{2}{(1-\beta)^2} \left\{\beta \ln\beta + (1-\beta)\ln(1-\beta) + \ln\frac{1+r}{2-\beta} - (1-\beta)\ln 2\right\}.$$

Provided that the outstanding local bonds are given, in each period, the consumption of private goods must equal that of local public goods for the temporal utility to be maximized. Under the MPE, this condition is satisfied in each period. In this sense, the intratemporal resource allocation is efficient under the MPE.

Let $\hat{x}_t$ and x_t^* denote the gross outstanding local bonds in period t corresponding to the MPE and to the optimal solution for the same (x_0^1, x_0^2), respectively. One can verify that $\hat{x}_t, x_t^* \to \frac{2}{r}$ as $t \to \infty$ and that, if $x_0^1 + x_0^2 < \frac{2}{r}$, $\hat{x}_t > x_t^*$ for all t since $\frac{\beta(1+r)}{2-\beta} < \beta(1+r)$. That is, the gross outstanding local bonds under this MPE are too high compared with the optimal solution. This implies overconsumption during the early periods and underconsumption in the succeeding periods.

2.4 Direct Overcompetitive Distortion

Under the MPE, the intratemporal resource allocation is efficient in the sense that as the consumption of private goods equals that of the local public goods in each period. However, the outcome of the MPE is inefficient since the intertemporal resource allocation is inefficient.

The reason for the distortion of intertemporal resource allocation may be explained as follows: If the interest rates are constant (r), an economic agent in the private sector who earns one unit of income in every period cannot borrow such that the outstanding amount is more than $\frac{1}{r}$. That is, the outstanding amount is constrained by the present value of future incomes ($\frac{1}{r}$). In contrast, in the case of local bonds backed by, or believed to be backed by, the central government, the sum of outstanding local bonds is constrained by the discounted sum of the future income in both regions. In other words, the upper bound of borrowing is common between agents. For example, each agent can borrow even if its outstanding amount is over the discounted sum of the future incomes, as long as the sum of the outstanding amounts is smaller than the discounted sum of the future incomes in both regions. In this case, the agents compete with each other in borrowing. This competition leads to too great a quantity of outstanding local bonds and distorts the intertemporal resource allocation.

The competition is shown in (14). The last term on the right-hand side of (14) represents that the higher the gross lifetime income is, the more each local government issues bonds. For example, a decrease in x_0^j causes a corresponding increase in the outstanding local bonds and a reduction in the local tax in region i as a result of the decrease in the gross lifetime income. We call this distortion induced through a common upper bound of borrowing *direct overcompetitive distortion*.

This borrowing competition is interpreted as the follows: Both local governments all together consume the gross lifetime income. This is because $x_t^1 + x_t^2 \leq \frac{2}{r}$ is equivalent to $c_t^1 + g_t^1 + c_t^2 + g_t^2 \leq I(x_{t-1}^1, x_{t-1}^2)$. By interpreting the gross lifetime income as a common resource, this game has the same structure as a social dilemma.

Moreover, provided x_{t-1}^i and z_t^i are given, each resident can attain an arbitrary amount of consumption (c_t^i, g_t^i) by setting the outstanding local bond (x_t^i) and local tax (y_t^i) as follows:

$$x_t^i = g_t^i + c_t^i - 1 - z_t^i + (1+r)x_{t-1}^i, \quad y_t^i = 1 - c_t^i + z_t^i.$$

In other words, local governments can decide the amount of consumption (c_t^i, g_t^i) freely, as long as they satisfy $c_t^1 + g_t^1 + c_t^2 + g_t^2 \leq I(x_{t-1}^1, x_{t-1}^2)$.[7] Essentially, this is the same structure as the model of Levhari and Mirman (1980).[8]

When the gross lifetime income increases, local governments try to increase the current consumption. For this purpose, each local government raises the net amount of bond issuance and, at the same time, decreases the local tax to achieve an intratemporal balance. By a transformation of equation (15), $x_t^1 + x_t^2 - (1 + r)(x_{t-1}^1 + x_{t-1}^2) = \frac{2}{2-\beta}(1-\beta)I(x_{t-1}^1, x_{t-1}^2) - 2$ follows. This equation together with equation (16) implies the abovementioned characteristics of the MPE. The solution of the planning problem has the same characteristics. However, from $\frac{2}{2-\beta} > 1$, the increase in the net amount of bond issuance in the MPE is greater than that in the planning problem when the gross lifetime income increases. From (11) and (16), the decrease in the local tax in the MPE is less than that in the planning problem when the gross lifetime income increases, since $\frac{1}{2-\beta} > \frac{1-\beta}{2}$. The larger amount of government bond issuance causes greater local tax reduction for intratemporal balance.

Efficiency may be improved if the central government commits itself to neglecting the redemption of local bonds. Suppose that the central government commits itself to neglecting the redemption of local bonds and that constraint (3) is replaced by $x_t^1 \leq \frac{1}{r}$, $x_t^2 \leq \frac{1}{r}$ for arbitrary $t \in \mathcal{T}$. The corresponding MPEs consist only of a pair of strategies whose outcomes are efficient, though the outcome does not always maximize social welfare.

3 Infinitely Iterated DL Model

The infinitely iterated DL model (or "DL model" for short) represents the following situation: In each year, local governments decide the outstanding local bonds and local taxes before the central government decides its subsidies to the resident in each region; and this process is repeated over many years.

[7]For this reason, it is inferred that the local governments' strategies are not controllable by the central government's strategies in any subgame perfect equilibrium.

[8]The consumption in Corollary 1 is the same as that in the MPE of the modified model of Levhari and Mirman (1980).

3.1 Definition of the Game and the Equilibrium

Environment The basic economic environment is the same as that in the previous section except in the stage game that takes place in each period. In each period, the players' decisions in the stage game are as follows:

(1) Both local governments decide outstanding local bonds x_t^i and local taxes y_t^i simultaneously. (First move in period t.)

(2) The central government decides subsidies (z_t^1, z_t^2) to each resident to satisfy $z_t^1 + z_t^2 = 0$. (Second move in period t.)

(3) Private goods consumption c_t^i and local public goods supply g_t^i are realized, where c_t^i and g_t^i satisfy (1) and (2), respectively.

Strategy Set Define the sets of histories ($\mathcal{H}_t^L$) up to the first move in period t as follows:

$$\mathcal{H}_t^L \equiv \begin{cases} \mathcal{B}_1 \times \mathcal{Q}_1 & \text{if } t = 1 \\ \mathcal{B}_t \times \mathcal{Q}_t \times \mathcal{S}_{t-1} & \text{if } t \in \mathcal{T}/\{1\}. \end{cases}$$

Using this notation and that defined in the previous section, the local governments' strategy sets $\mathcal{A}^{D1}$ and $\mathcal{A}^{D2}$ and the central government's strategy set $\mathcal{A}^{D0}$ are defined as follows: For $i = 1, 2$,

$$\begin{aligned} \mathcal{A}^{Di} \equiv & \Big\{((b_1, q_1), (b_2, q_2), \cdots) \in \Pi_{t\in\mathcal{T}} \left(\mathbb{R}^{\mathcal{H}_{t-1}}\right)^2 \Big| (\forall t \in \mathcal{T})(\forall h \in \mathcal{H}_{t-1}) \\ & q_t(h) + b_t(h) - (1+r)x_{t-1}^i \geq 0 \ where \\ & h = ((x_0^1, \cdots, x_{t-1}^1), (x_0^2, \cdots, x_{t-1}^2), \mathbf{y}_{t-1}, \mathbf{z}_{t-1})\Big\}, \\ \mathcal{A}^{D0} \equiv & \Big\{((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots) \in \Pi_{t\in\mathcal{T}}(\mathbb{R}^{\mathcal{H}_t^L})^2 \Big| (\forall t \in \mathcal{T})(\forall h \in \mathcal{H}_t^L) \\ & (\forall i \in \{1, 2\}) \ 1 - y_t^i + s_t^i(h) \geq 0, \ s_t^1(h) + s_t^2(h) = 0 \ where \\ & h = (\mathbf{x}_t, ((y_1^1, \cdots, y_t^1), (y_1^2, \cdots, y_t^2)), \mathbf{z}_{t-1})\Big\}. \end{aligned}$$

For given $((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots) \in \mathcal{A}^{D0}$, $s_t^i \in \mathbb{R}^{\mathcal{H}_t^L}$ represents a function that assigns subsidies to the resident in region i in period t. In period t, the central government decides subsidies depending on the history up to period $(t-1)$ and also based on the local governments' decisions in period t. On the other hand, for given $((b_1, q_1), (b_2, q_2), \cdots) \in \mathcal{A}^{D1}$, $b_t \in \mathbb{R}^{\mathcal{H}_{t-1}}$ ($q_t \in \mathbb{R}^{\mathcal{H}_{t-1}}$) represents a function that assigns outstanding local bonds (local taxes) in period t. In period t, the local governments decide the outstanding local bonds and local taxes depending on the history from the first period to period $(t-1)$.

However, for some $t \in \mathcal{T}$, $((b_1^1, q_1^1), (b_2^1, q_2^1), \cdots) \in \mathcal{A}^{D1}$, $((b_1^2, q_1^2), (b_2^2, q_2^2), \cdots) \in \mathcal{A}^{D2}$, and $h \in \mathcal{H}_{t-1}$, it may be that $b_t^1(h) + b_t^2(h) > \frac{2}{r}$ or that $q_t^1(h) + q_t^2(h) > 2$. In such a case, the outstanding local bonds determined by b_t^1, b_t^2 violate (3) or the sum of the local tax determined from q_t^1 and q_t^2 violate (4). Therefore, if $b_t^1(h) + b_t^2(h) > \frac{2}{r}$, the outstanding local bonds (x_t^i) and local taxes (y_t^i) of local government i in period t are assumed to be

$$x_t^1 = x_t^2 = \frac{1}{r},\ y_t^i = (1+r)x_{t-1}^i - \frac{1}{r}, \tag{17}$$

and, if $q_t^1(h) + q_t^2(h) > 2$ and $b_t^1(h) + b_t^2(h) \leq \frac{2}{r}$, they are assumed to be[9]

$$x_t^i = (1+r)x_{t-1}^i - 1, \quad y_t^i = 1. \tag{18}$$

Definition of Equilibria Let $\mathcal{A}^D$ denote $\mathcal{A}^{D0} \times \mathcal{A}^{D1} \times \mathcal{A}^{D2}$. When a combination of strategies, $\mathbf{a} = (a^0, a^1, a^2) \in \mathcal{A}^D$, and the history up to a period τ, $h \in \mathcal{H}_\tau$, are given, the sequence of consumption in region i after period $(\tau+1)$, $\mathbf{w}^i(\mathbf{a}, h) = (\{c_t^i\}_{t\in\mathcal{T}}, \{g_t^i\}_{t\in\mathcal{T}})$, can be determined uniquely for $i = 1, 2$. For example, when $a^0 = ((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots)$, $a^i = ((b_1^i, q_1^i), (b_2^i, q_2^i), \cdots)$, $h = ((\mathbf{x}_\tau^1, \mathbf{x}_\tau^2), (\mathbf{y}_\tau^1, \mathbf{y}_\tau^2), \mathbf{z}_\tau)$, and $\mathbf{x}_\tau^1 = (x_0^i, \cdots, x_\tau^i)$, c_t^i and g_t^i are determined by (1) and (2), where $x_0^i = x_\tau^i$; and for $t \in \mathcal{T}$ at which $b_{\tau+t}^1 + b_{\tau+t}^2 \leq \frac{2}{r}$ and $q_{\tau+t}^1 + q_{\tau+t}^2 \leq 2$, x_t^i, y_t^i, and z_t^i are determined in the following manner:[10]

$$x_t^i = b_{\tau+t}^i(h_{t-1}), \quad y_t^i = q_{\tau+t}^i(h_{t-1}), \quad \mathbf{x}_{\tau+t}^i = (\mathbf{x}_{\tau+t-1}^i, x_t^i), \quad \mathbf{y}_{\tau+t}^i = (\mathbf{y}_{\tau+t-1}^i, y_t^i),$$

[9] Under this determination of x_t^i and y_t^i, $x_t^1 + x_t^2 \leq \frac{2}{r}$, $y_t^1 + y_t^2 \leq 2$, $y_t^i + x_t^i - (1+r)x_{t-1}^i \geq 0$, and $(\frac{2}{r} - x_t^1 - x_t^2)(2 - y_t^1 - y_t^2) = 0$. The equilibrium strategy presented in Proposition 3 necessarily satisfies $b_t^1(h) + b_t^2(h) \leq \frac{2}{r}$ and $q_t^1(h) + q_t^2(h) \leq 2$. The result does not depend on the method used to determine x_t^i and y_t^i when $b_t^1(h) + b_t^2(h) > \frac{2}{r}$ or $q_t^1(h) + q_t^2(h) > 2$, as long as they satisfy the conditions mentioned above.

[10] If $b_{\tau+t}^1 + b_{\tau+t}^2 > \frac{2}{r}$, x_t^i and y_t^i are determined through (17). If $b_{\tau+t}^1 + b_{\tau+t}^2 \leq \frac{2}{r}$ and $q_{\tau+t}^1 + q_{\tau+t}^2 > 2$, x_t^i and y_t^i are determined through (18).

$$h_0 = h, \quad z_t^i = s_\tau^i(\mathbf{x}_{\tau+t}^1,\, \mathbf{x}_{\tau+t}^2,\, \mathbf{y}_{\tau+t}^1,\, \mathbf{y}_{\tau+t}^2,\, \mathbf{z}_{\tau+t-1}),$$

$$\mathbf{z}_{\tau+t}^i = (\mathbf{z}_{\tau+t-1}^i, (z_t^1, z_t^2)), \quad h_t = (\mathbf{x}_{\tau+t}^1,\, \mathbf{x}_{\tau+t}^2,\, \mathbf{y}_{\tau+t}^1,\, \mathbf{y}_{\tau+t}^2,\, \mathbf{z}_{\tau+t}).$$

Similarly, when a combination of strategies, $\mathbf{a} \in \mathcal{A}^D$, and the history up to the first move in period t, $h \in \mathcal{H}_t^L$, are given, the sequence of consumption after period t, $\mathbf{w}^i(\mathbf{a}, h)$, can be determined uniquely.

In particular, for $(x_0^1, x_1^1, x_0^2, x_1^2, y_1^1, y_1^2) \in \mathcal{H}_1^L$, both $\mathbf{w}^i(\mathbf{a}, (x_0^1, x_1^1, x_0^2, x_1^2, y_1^1, y_1^2))$ and $\mathbf{w}^i(\mathbf{a}, (x_0^1, x_0^2))$ represent a sequence of consumption from period 1; moreover, they are identical sequences if $x_1^i = b_1^i(x_0^1, x_0^2)$ and $y_1^i = q_1^i(x_0^1, x_0^2)$; that is,

$$\mathbf{w}^i(\mathbf{a}, (x_0^1, x_0^2)) = \mathbf{w}^i(\mathbf{a}, (x_0^1, b_1^1(x_0^1, x_0^2), x_0^2, b_1^2(x_0^1, x_0^2), q_1^1(x_0^1, x_0^2), q_1^1(x_0^1, x_0^2))).$$

Definition 3 A combination of strategies $(a^{0*}, a^{1*}, a^{2*}) \in \mathcal{A}^D$ is a subgame perfect equilibrium of the DL model if it satisfies the following conditions.

$$(\forall t \in \mathcal{T})\, (\forall h \in \mathcal{H}_t^L) \;\; a^{0*} \in \arg\max_{a \in \mathcal{A}^{D0}} \sum_{i=1,2} \theta^i U^i(\mathbf{w}^i((a, a^{1*}, a^{2*}), h)),$$

$$(\forall t \in \mathcal{T})\, (\forall h \in \mathcal{H}_{t-1}) \;\; a^{1*} \in \arg\max_{a \in \mathcal{A}^{D1}} U^1(\mathbf{w}^1((a^{0*}, a, a^{2*}), h)),$$

$$(\forall t \in \mathcal{T})\, (\forall h \in \mathcal{H}_{t-1}) \;\; a^{2*} \in \arg\max_{a \in \mathcal{A}^{D2}} U^2(\mathbf{w}^2((a^{0*}, a^{1*}, a), h)).$$

Furthermore, an MPE is defined. When functions $f \in \mathbb{R}^{\mathcal{H}_0}$ and $e \in \mathbb{R}^{\mathcal{H}_1^L}$ are given, the functions $\mathbf{f} = (f_1, f_2, \cdots) \in \Pi_{t \in \mathcal{T}} \mathbb{R}^{\mathcal{H}_{t-1}}$ and $\mathbf{e} = (e_1, e_2, \cdots) \in \Pi_{t \in \mathcal{T}} \mathbb{R}^{\mathcal{H}_t^L}$ can be determined uniquely in the following manner:

$$f_1 = f \quad \text{and} \quad e_1 = e, \tag{19}$$

$$(\forall t \in \mathcal{T}) \;\; f_{t+1}(\mathbf{x}_t^1,\, \mathbf{x}_t^2,\, \mathbf{y}_t^1,\, \mathbf{y}_t^2,\, \mathbf{z}_t) = f(x_t^1, x_t^2), \tag{20}$$

$$(\forall t \in \mathcal{T}) \;\; e_{t+1}(\mathbf{x}_{t+1}^1,\, \mathbf{x}_{t+1}^2,\, \mathbf{y}_{t+1}^1,\, \mathbf{y}_{t+1}^2,\, \mathbf{z}_t) = e(x_t^1, x_{t+1}^1, x_t^2, x_{t+1}^2, y_{t+1}^1, y_{t+1}^2), \tag{21}$$

where $(\mathbf{x}_t^1,\, \mathbf{x}_t^2) \in \mathcal{B}_t$, $\mathbf{x}_t^i = (x_0^i, x_1^i, \cdots, x_t^i)$, $(\mathbf{y}_t^1,\, \mathbf{y}_t^2) \in \mathcal{Q}_t$, $\mathbf{y}_t^i = (y_1^i, y_2^i, \cdots, y_t^i)$, and $\mathbf{z}_t \in \mathcal{S}_t$. Using (19) and (20), for $i = 1, 2$, we can construct a strategy $a \in \mathcal{A}^{Di}$ from a combination of functions $(e, f) \in (\mathbb{R}^{\mathcal{H}_0})^2$ if $q(h) + b(h) - (1+r)x^i \geq 0$, where $h = (x^1, x^2)$. Hence, we state that a combination of functions $(b, q) \in (\mathbb{R}^{\mathcal{H}_0})^2$ satisfying $q(h) + b(h) - (1+r)x^i \geq 0$ is a Markov strategy of local government i.

Similarly, by using (19) and (21), we can construct a strategy $a \in \mathcal{A}^{D0}$ from a combination of functions $(e, f) \in (\mathbb{R}^{\mathcal{H}_1^L})^2$ if $e + f = 0$. Hence, we state that a combination of functions $(s^1, s^2) \in (\mathbb{R}^{\mathcal{H}_1^L})^2$ satisfying $s^1 + s^2 = 0$ is a Markov strategy of the central government.

Definition 4 A combination of Markov strategies $((s^1, s^2), (b^1, q^1), (b^2, q^2))$ is an MPE of the DL model if $(a^0, a^1, a^2) \in \mathcal{A}^D$ is a subgame perfect equilibrium, where a^0 is constructed from (s^1, s^2) by using (19) and (21) and, for $i = 1, 2$, a^i is constructed from (b^i, q^i) by using (19) and (20).

3.2 Markov Perfect Equilibrium

Proposition 3 $((\tilde{s}^1, \tilde{s}^2), (\tilde{b}^1, \tilde{q}^1), (\tilde{b}^2, \tilde{q}^2))$ is an MPE of the DL model if for $(x^1, x^2) \in \mathcal{H}_0$ and $((x_0^1, x_1^1), (x_0^2, x_1^2), y^1, y^2) \in \mathcal{H}_1^L$,

$$(\forall i = 1, 2) \;\; \tilde{s}^i((x_0^1, x_1^1), (x_0^2, x_1^2), y^1, y^2) = \frac{y^i - y^j}{2} \quad (j \neq i),$$

$$(\forall i = 1, 2) \;\; \tilde{q}^i(x^1, x^2) + \tilde{b}^i(x^1, x^2) = \frac{1 - \beta}{3 - \beta} I(x^1, x^2) + (1 + r)x^i, \tag{22}$$

$$\tilde{b}^1(x^1, x^2) + \tilde{b}^2(x^1, x^2) = \frac{2}{r} - \frac{2\beta}{3 - \beta} I(x^1, x^2). \tag{23}$$

Functional equations for the MPE, corresponding to the Bellman equation of a dynamic programming are as follows for $i = 1$, 2:

$$V^i(x^1, x^2) = F^i((x^1, b^1(x^1, x^2)), (x^2, b^2(x^1, x^2)), q^1(x^1, x^2), q^2(x^1, x^2)),$$

$$F^i(h) = \ln(1 - y^i + s^i(h)) + \ln(y^i + x_1^i - (1 + r)x_0^i) + \beta V^i(x_1^1, x_1^2),$$

$$V^1(x^1, x^2) = \max_{x,y} F^1((x^1, x), (x^2, b^2(x^1, x^2)), y, q^2(x^1, x^2)),$$

$$V^2(x^1, x^2) = \max_{x,y} F^2((x^1, b^1(x^1, x^2)), (x^2, x), q^1(x^1, x^2), y),$$

$$\sum_{i=1}^{2} F^i(h) = \max_{(z^1, z^2) \in \mathcal{S}_1} \sum_{i=1}^{2} \{\ln(1 - y^i + z^i) + \ln(y^i + x_1^i - (1 + r)x_0^i) + \beta V^i(x_1^1, x_1^2)\},$$

where $h = ((x_0^1, x_1^1), (x_0^2, x_1^2), y^1, y^2) \in \mathcal{H}_1^L$ and $((s^1, s^2), (b^1, q^1), (b^2, q^2))$ is a combination of Markov strategies. The functional equations are satisfied by $V^1 = V^2 = V^D$ (V^D will be presented in Corollary 2) and $((\tilde{s}^1, \tilde{s}^2), (\tilde{b}^1, \tilde{q}^1), (\tilde{b}^2, \tilde{q}^2))$ in

Proposition 3. However, a proof for the Proposition is necessary, since we cannot directly apply the theory of dynamic programming. See Appendix C for proof.

The central government allocates subsidies to the region imposing higher local tax so that the private goods consumption in both regions is the same. This will provide the local governments with an incentive to impose heavier taxes.

There exist uncountably many MPEs satisfying (22) and (23). Consequently, it is difficult to characterize the MPE strategies of local governments. Fortunately, we can derive a unique consumption path from the strategies in Proposition 3, as in Corollary 1. Corollary 2 is a description of the consumption path, outstanding local bonds, and resident's utility in the DL model.

Corollary 2 The gross outstanding local bonds, private consumption, and public goods supply corresponding to the x_t^i, y_t^i, z_t^i values determined by all the strategies in Proposition 3 are the same. Furthermore, they satisfy the following equations:

$$I(x_t^1,\, x_t^2) = \frac{2\beta(1+r)}{3-\beta} I(x_{t-1}^1,\, x_{t-1}^2), \tag{24}$$

$$y_t^1 + y_t^2 = 2 - \frac{1-\beta}{3-\beta}\, I(x_{t-1}^1,\, x_{t-1}^2), \tag{25}$$

$$2c_t^1 = 2c_t^2 = g_t^1 = g_t^2 = \frac{1-\beta}{3-\beta}\, I(x_{t-1}^1,\, x_{t-1}^2),$$

$$U^1(\{c_t^1, g_t^1\}_{t\in\mathcal{T}}) = U^2(\{c_t^2, g_t^2\}_{t\in\mathcal{T}}) = V^D(x_0^1, x_0^2),$$

$$V^D(x_0^1, x_0^2) = \frac{2}{1-\beta} \ln\left(\frac{2}{r} - x_0^1 - x_0^2\right) + \delta^D,$$

$$\delta^D = \frac{2}{(1-\beta)^2}\ \left\{\beta\ln\beta + (1-\beta)\ln(1-\beta) + \ln\frac{1+r}{3-\beta} + \frac{3\beta-1}{2}\ln 2\right\}.$$

Provided the outstanding local bonds are given, in each period, the consumption of private goods must equal that of local public goods for the temporal utility to be maximized. However, private goods are consumed only half as much as local public goods in each period under this MPE. In this sense, the intratemporal resource allocation is inefficient under the MPE.

Let $\hat{x}_t$, $\tilde{x}_t$, and x_t^* denote the gross outstanding local bonds in period t corresponding to the MPE in Proposition 2, the MPE in Proposition 3, and the optimal solution for the same (x_0^1, x_0^2), respectively. One can verify that $\tilde{x}_t, x_t^*, \hat{x}_t \to \frac{2}{r}$ as

$t \to \infty$ and that, if $x_0^1 + x_0^2 < \frac{2}{r}$, $\tilde{x}_t > x_t^*$ for all t since $\frac{2\beta(1+r)}{3-\beta} < \beta(1+r)$. That is, the gross outstanding local bonds under the MPE in Proposition 3 are too high compared with the optimal solution. This implies overconsumption during the early periods and underconsumption in the succeeding periods.

Furthermore, if $x_0^1 + x_0^2 < \frac{2}{r}$, $\tilde{x}_t < \hat{x}_t$ for all t since $\frac{\beta(1+r)}{2-\beta} < \frac{2\beta(1+r)}{3-\beta}$. (See equations (15) and (24).) This implies that the distortion of the intertemporal resource allocation under the MPE of the DL model given in Proposition 3 is less aggravated than that of the CL model given in Proposition 2. This point is discussed further in Section 5.

3.3 Basic SBC Distortion and Price Effect Distortion

The outcome of the MPE in Proposition 3 is inefficient. Furthermore, under the MPE, both intra- and intertemporal resource allocations are inefficient. The reason for the intratemporal inefficiency is explained as follows: The central government allocates subsidies to the region imposing higher local tax so that the private goods consumption in both regions is the same. This gives local governments an incentive to impose heavier taxes. That is, the subsidy policy of the central government distorts the intratemporal resource allocation. We call the distortion that is induced by the central government's actions in the same period as the action itself *basic SBC distortion*.

On the other hand, the reason for the distortion of the intertemporal resource allocation may be explained as follows: If a local government expands its outstanding local bonds in period $(t-1)$, it raises the local taxes to redeem the local bonds or to paying the interest in period t. This increase in local taxes reduces private goods consumption. This reduction in private goods consumption is the cost of expanding the outstanding local bonds. However, the cost is reduced because the central government provides subsidies to the region where local taxes are high.[11] In this manner, the subsidy policy of the central government in period t gives local governments an incentive to expand their outstanding local bonds in period $(t-1)$. We call the distortion that is caused by local governments' expec-

[11] See Goodspeed (2002).

tation of central-government-sponsored bailouts in subsequent periods *price effect distortion*.

Local governments try to increase the amount of current consumption when the gross lifetime income increases. For this purpose, each local government raises the net amount of bond issuance and, at the same time, decreases the local tax for intratemporal balance. From equation (15), $x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2) = \frac{2}{2-\beta}(1-\beta)I(x_{t-1}^1, x_{t-1}^2) - 2$. This equation and equation (16) imply the above character of the MPE. Moreover, from $\frac{3}{3-\beta} > 1$, the increase in the net amount of bond issuance in the MPE is more than that in the planning problem when the gross lifetime income increases. However, from equations (11) and (25), the decrease in the local taxes in the MPE is less than that in the planning problem when the gross lifetime income increases since $\frac{1}{3-\beta} < \frac{1-\beta}{2}$. One might infer that the larger amount of government bond issuance cause the greater local tax reduction for intratemporal balance; however, this inference is wrong because the subsidy provides local governments with an incentive to increase local taxes or inhibit tax cuts.

4 Finite-period Models

In Propositions 2 and 3, we provide the MPE of the CL and DL model respectively. However, uniqueness problem remains an open question.[12] In this section, we attempt to provide some justification.

Concretely, we proceed to a discussion as follows. First, we explain that in the finite models that converge to the infinitely iterated models as time tends to infinity, the consumption paths and the value function realized under any subgame perfect equilibrium are unique. Second, we assert that the consumption paths and the value function given in Corollaries 1 and 2 are the limits of the consumption paths and the value functions in the finite-period models; that is, some kind of continuity is preserved.

[12] Tsutsui and Mino (1990) has investigated non-linear Markov equilibria in a linear-quadratic differential game.

4.1 T-period Model

T-period CL and DL models are almost identical to the CL and DL models, respectively, except that the economic agents live for T periods. The utility function of the representative residents is modified as

$$\sum_{t=1}^{T} \beta^{t-1}\{\ln c_t^i + \ln g_t^i\}.$$

For $t = 1, \cdots, T$, we define the net present value of the lifetime income in both regions in period t by

$$I_t(x_{t-1}^1, x_{t-1}^2) = \sum_{\tau=t}^{T} \frac{2}{(1+r)^{\tau-t}} + \frac{\bar{x}^1 + \bar{x}^2}{(1+r)^{T-t}} - (1+r)(x_{t-1}^1 + x_{t-1}^2).$$

Equation (3), the constraint of outstanding local bonds, is modified as

$$(\forall i = 1, 2)\ (\exists \bar{x}^i \geq 0)\ \ x_T^i \leq \bar{x}^i.$$

Note that $\lim_{T\to\infty} I_t(x^1, x^2) = I(x^1, x^2)$ for all $t \in \mathcal{T}$ and for all $(x^1, x^2) \in \mathcal{H}_0$. Therefore, the CL (DL) model is the limit of the finite-period CL (DL) model as $T \to \infty$.

Proposition 4 For arbitrary T and for an arbitrary subgame perfect equilibrium of a T-period CL (DL) model under which the utility of all the residents is finite, the corresponding consumption path and value function are the same. Furthermore, the consumption paths and the value function realized under the MPE of Proposition 2 (Proposition 3) are the limits of the sequence of consumption paths and the sequence of the value function as $T \to \infty$, respectively.

The subgame perfect equilibrium as well as the corresponding consumption path and value function are given in Appendix D.

4.2 Reason for Focusing on Infinitely Iterated Models

Infinitely iterated models are the limits of finite-period models; this may suggest that the finite-period models are sufficient for analysis. However, the infinitely iterated models are necessary due to the following reason:

In case any one of the players is a government entity, as in the present paper, the existence of a last period is inappropriate as a government entity does not disappear. In addition, the number of periods considered in a model, viz., a finite or infinite number, may cause a large variation in the outcome as in a repeated prisoner's dilemma since agents would behave strategically in the last period if the model is of a finite period.

Furthermore, in the finite-period models, it is difficult to identify the implications of equilibrium strategies because they depend on specific time coordinates. On the other hand, in the infinitely iterated models, it is relatively simpler to interpret the strategies of the MPEs because they are independent of time.

Based on the above discussion, we focus on the MPEs of Propositions 2 and 3 in the following section.

5 Comparison between the CL and DL models

5.1 Social Welfare

Table 1 summarizes Sections 2 and 3. Unlike in the CL model, the intratemporal resource allocation in the MPE is inefficient in the DL model. On the one hand, the intertemporal resource allocation in the MPE is inefficient both in the CL and DL model; and on the other hand, the intertemporal resource allocation in the MPE in the DL model is more efficient than in the DL model. We therefore cannot judge which allocation is more efficient overall — the CL model or the DL model. Consequently, we resort to comparisons of the social welfare in the two models.

	Intratemporal Distortion	Intertemporal Distortion
CL	nonexistence	Direct Overcompetitive Distortion
DL	Basic SBC Distortion	Price Effect Distortion

Table 1: Comparison of distortion

A difference between the social welfare attained by the outcome of the MPE of the DL model given in Proposition 3 and that of the CL model given in Proposition 2 is

$$2\{V^D(x^1,x^2) - V^C(x^1,x^2)\} = \frac{2}{(1-\beta)^2}\left\{(1+\beta)\ln 2 - 2\ln\frac{3-\beta}{2-\beta}\right\}.$$

The difference depends only on β. Let $\Delta(\beta)$ denote the right hand side of the equation. It is simple to see that $\Delta'(\beta) > 0$, $\Delta(0) < 0$, and $\lim_{\beta\to 1}\Delta(\beta) = \infty$. Hence, there exists some value $\bar{\beta}$ such that $\Delta(\bar{\beta}) = 0$ uniquely. Social welfare in the DL (CL) model is more than that in the CL (DL) model when $\beta > (<)\bar{\beta}$. For plausible values of β, the DL model is more desirable than the CL model from the viewpoint of social welfare.[13]

In the DL model, the intratemporal and intertemporal resource allocations are both distorted. On the other hand, only the intertemporal resource allocation is distorted in the CL model. However, the total distortion in the CL model is more aggravated than that in the DL model if β is sufficiently large; this stems from the fact that the larger is the value of β, the more important is future consumption compared to present consumption and, furthermore, direct overcompetitive distortion is more aggravated than price-effect distortion. Why is direct overcompetitive distortion more aggravated than price effect distortion? This point will be discussed in Section 5.2.

In addition, in finite models, it may hold that the social welfare in the DL model is greater than that in the CL model if β is sufficiently large. First, when $T = 2$, the CL model is more desirable than the DL model for all $\beta \in (0,1)$. On the other hand, for $T \geq 3$, the DL model is more desirable than the CL model if β is sufficiently large. This shows that the analysis with two-period models leads to a conclusion contrary to that with any other finite-period model and the infinitely iterated models.

Furthermore, $\bar{\beta}_{T+1}$ is smaller than $\bar{\beta}_T$, where $\bar{\beta}_T$ represents the value of β at which the social welfare of the T-period DL and CL models are equal.[14] In other words, the set of values of β for which a DL model is more desirable than a CL model is expanded.

[13] β is about 0.4.

[14] It is trivial that $\bar{\beta}_T \to \bar{\beta}$ as $T \to \infty$.

5.2 Mechanisms of Intertemporal Distortion

As described in Section 2.4, both local governments under any equilibrium consume the gross lifetime income in the CL model all together. By interpreting the gross lifetime income as a common resource, this game has the same structure as a kind of social dilemma and, furthermore, overconsumption occurs in these games; however, in the DL model as well as in the CL model, both local governments scramble to consume the gross lifetime income. What causes this difference in outcome between the CL and DL model? As a matter of fact, each model has a different route for the emergence of distortion. In what follows, we will attempt an explanation of this difference.

First, in the DL model, the local governments cannot always decide the amount of consumption freely, while they can by setting the bond issuance and local tax appropriately in the CL model. As explained in Section 2.3, in the CL model, the local governments compete with each other in consuming the common resources since the local governments' strategies are not controllable by the central government's strategy. On the other hand, the local governments may not be able to decide c_t^i and g_t^i freely, depending on central government's strategies in the DL model. Suppose, for example, that the central government allocates a subsidy (z_t^i) to region i, $z_t^i = \frac{y_t^i - y_t^j}{2}$. Then, the private consumption is the same in both regions, regardless of the actions of the local governments. In other words, the competition between the local governments can be restricted.

Second, in the DL model, both local governments mainly attempt to deprive each other of resources through the subsidy; while, in the CL model, each local government consumes excessively at the expense of future resources through outstanding local bonds. As explained in Section 3.3, in the DL model, the local governments have an incentive to issue bonds excessively. However, in the DL model, even if a local government attempts to increase the present consumption at the expense of future resources through outstanding local bonds as in the CL model, the local government may be prevented from increasing the present consumption by the other local government through subsidy. For this reason, although there is intertemporal distortion in both models, the price-effect distortion in the

DL model is not as aggravated as the direct overcompetitive distortion in the CL model.

From the above discussion, we can state that if β is sufficiently large, the DL model is more desirable than the CL model since the competitive restriction in the DL model weakens the degree of intertemporal distortion.

5.3 Value of Delay in Subsidization

We define functions F^C and F^D as below and define α as the value satisfying the following equation. For $i = C, D$,

$$\begin{aligned} F^i(x) &= \frac{2}{1-\beta}\ln x + \delta^i - \frac{2}{1-\beta}\ln(1+r), \\ F^D(x) &= F^C((1+\alpha)x). \end{aligned}$$

Since $F^C(I(x^1, x^2)) = V^C(x^1, x^2)$ and $F^D(I(x^1, x^2)) = V^D(x^1, x^2)$, for given r and β, F^C and F^D can be considered as functions that assign utility corresponding to the MPE in Propositions 2 and 3 when the gross lifetime income is given, respectively. Hence, in the CL model, α represents the requisite percentage change in the gross lifetime income in order to attain the same social welfare as in the DL model. In other words, α is the value of the delay in subsidization measured in terms of the gross lifetime income; it is simple to see that $\alpha = \exp(\frac{(1-\beta)}{4}\Delta(\beta)) - 1$.

If β is sufficiently close to unity, the value of the delay in subsidization is nearly equal to 16% of the gross lifetime income.

As for finite-period models, we can see the value of the delay in subsidization. One can verify that, for each β, the longer is the time span T, the higher are the values of the delay in subsidization.[15]

[15]It is trivial that the value of the delay in subsidization converges to α as T tends to ∞. When $T = 4$, the values of the delay in subsidization are lower than 3% of the gross lifetime income. This value, 3%, is much lower than the value of the delay in subsidization for infinitely iterated models when β is close to 1. This implies that analysis with 3- or 4-period models leads to underestimation of the values of the delay in subsidization.

5.4 Consumption Paths

Local Public Goods In the first several periods, local public goods are supplied in excess in the CL and DL models from the standpoint of social welfare. In the succeeding periods, the public goods supply drops below the optimal level in the both models. This is because the gross outstanding local bonds under the MPEs both in Propositions 2 and 3 are too high from the standpoint of social welfare. However, the public goods supply in the DL model is greater than that in the CL model in every period because of basic SBC distortion. To comprehend the general features of the public goods consumption paths, an example of the paths for specific parameters may be useful. Figure 2 presents the local public goods paths for specific example parameters: number of periods = 70, $\beta = 0.9$, $r = 0.04$, $x_0^1 = x_0^2 = 0$.

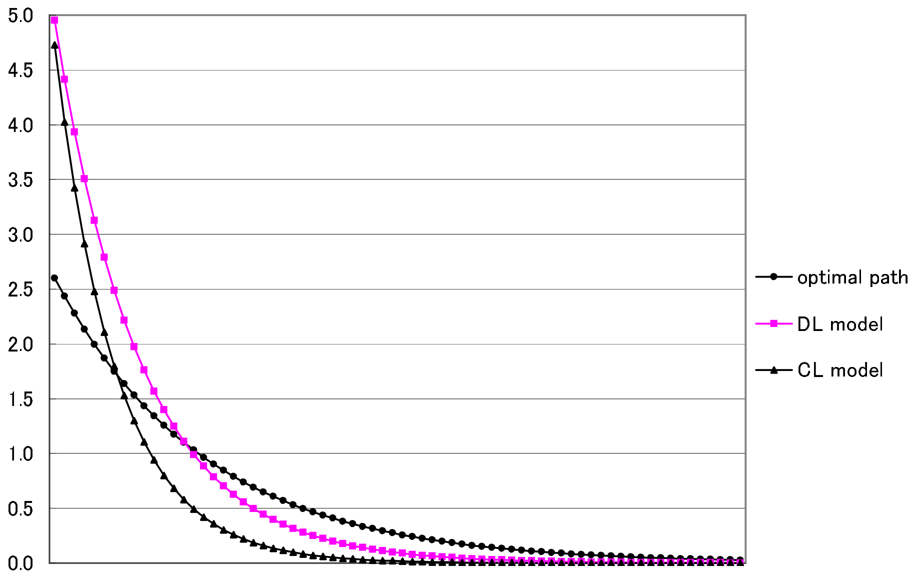


Figure 2: Example of local public goods paths

Private Goods The private goods consumption under the MPEs of the CL model in Proposition 2 is equal to the local public goods consumption in every period as well as that of the optimal path. On the other hand, in the DL model, private goods are consumed half as much as local public goods in each period because of basic SBC distortion. Additionally, the private goods consumption in the DL model is smaller than the optimal level in every period because of basic SBC distortion.

6 Conclusion

In Propositions 2 and 3, we provide the MPEs of the CL and DL models respectively. The outcomes of the MPEs in both models are inefficient. The root cause is the same, namely, both local governments scramble to consume the gross lifetime income. By interpreting the gross lifetime income as a common resource, this model has the same structure as a kind of social dilemma and, furthermore, overconsumption occurs in these models.

In the DL model, intra- and intertemporal resource allocation are both distorted; we refer to these phenomena as basic SBC distortion and price-effect distortion, respectively. On the other hand, only intertemporal resource allocation is distorted in the CL model, i.e., direct overcompetitive distortion occurs.

However, the total distortion in the CL model is more aggravated than that in the DL model if β is sufficiently large: by competitive restriction, direct overcompetitive distortion is more aggravated than price-effect distortion with respect to each transition from one period to the next, and furthermore, this difference is amplified if agents attach importance to future consumption.

It is an open question whether or not there exists an MPE that has a plausible consumption path different from those in Propositions 2 and 3; the matter remains to be proved. Unfortunately, there is an MPE that has an implausible consumption path: in the MPE, both local governments try to swallow all the resources in a fashion comparable to a swarm of locusts, i.e., $b_t^1 + b_t^2 \geq 2/r$ for any t.

Appendices

A Definition of the History Sets

The formal definitions of $\mathcal{B}$, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{B}_{t-1}$, $\mathcal{Q}_t$, and $\mathcal{S}_t$ are as follows.

$$\begin{aligned}
\mathcal{B} &\equiv \Big\{((x_0^1, x_1^1, \cdots), (x_0^2, x_1^2, \cdots)) \in (\mathbb{R}^{\mathcal{T}})^2 \Big| \ (\forall t \in \mathcal{T}) \ x_t^1 + x_t^2 \le \frac{2}{r} \ \text{ and} \\
&\qquad x_t^1 + x_t^2 = \frac{2}{r} \text{ if } x_{t-1}^1 + x_{t-1}^2 = \frac{2}{r}\Big\}, \\
\mathcal{Q} &\equiv \Big\{((y_1^1, y_2^1, \cdots), (y_1^2, y_2^2, \cdots)) \in (\mathbb{R}^{\mathcal{T}})^2 \Big| \ (\forall t \in \mathcal{T}) \ y_t^1 + y_t^2 \le 2\Big\}, \\
\mathcal{S} &\equiv \Big\{((z_1^1, z_1^2), (z_2^1, z_2^2), \cdots) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \Big| \ (\forall t \in \mathcal{T}) \ z_t^1 + z_t^2 = 0\Big\}, \\
\mathcal{B}_{t-1} &\equiv \Big\{ (\mathbf{x}_{t-1}^1, \mathbf{x}_{t-1}^2) \in (\mathbb{R}^t)^2 \Big| \ (\exists\, (\mathbf{x}^1, \mathbf{x}^2) \in \mathcal{B}) \ ((\mathbf{x}_{t-1}^1, \mathbf{x}^1), (\mathbf{x}_{t-1}^2, \mathbf{x}^2)) \in \mathcal{B}\Big\}, \\
\mathcal{Q}_t &\equiv \Big\{ (\mathbf{y}_t^1, \mathbf{y}_t^2) \in (\mathbb{R}^t)^2 \Big| \ (\exists\, (\mathbf{y}^1, \mathbf{y}^2) \in \mathcal{B}) \ ((\mathbf{y}_t^1, \mathbf{y}^1), (\mathbf{y}_t^2, \mathbf{y}^2)) \in \mathcal{B}\Big\}, \\
\mathcal{S}_t &\equiv \Big\{ \mathbf{z}_t \in \left(\mathbb{R}^2\right)^t \Big| \ (\exists\, \mathbf{z} \in \mathcal{S}) \ (\mathbf{z}_t, \mathbf{z}) \in \mathcal{S}\Big\}.
\end{aligned}$$

B Proof of Proposition 2

For the proof, it is sufficient to show that the following conditions hold because of the recursivity of the model.

$$(\forall \mathbf{x} \in \mathcal{H}_0) \ \ \hat{a}^0 \in \arg\max_{a \in \mathcal{A}^{C0}} \sum_{i=1,2} U^i(\mathbf{w}^i((a, \hat{a}^1, \hat{a}^2), \mathbf{x})), \tag{26}$$

$$(\forall h \in \mathcal{H}_1^C) \ \ \hat{a}^1 \in \arg\max_{a \in \mathcal{A}^{C1}} U^1(\mathbf{w}^1((\hat{a}^0, a, \hat{a}^2), h)), \tag{27}$$

$$(\forall h \in \mathcal{H}_1^C) \ \ \hat{a}^2 \in \arg\max_{a \in \mathcal{A}^{C2}} U^2(\mathbf{w}^2((\hat{a}^0, \hat{a}^1, a), h)), \tag{28}$$

where $\hat{a}^0, \hat{a}^1, \hat{a}^2$ are the strategies constructed from the Markov strategies $(\hat{b}^1, \hat{q}^1)$, $(\hat{b}^2, \hat{q}^2)$, and $(\hat{s}^1, \hat{s}^2)$. Let $h = (x^1, x^2, z^1, z^2) \in \mathcal{H}_1^C$ and, further, let $\hat{\mathbf{a}}$ denote $(\hat{a}^0, \hat{a}^1, \hat{a}^2)$. For arbitrary $a \in \mathcal{A}^{C0}$, we can verify that $U^i(\mathbf{w}^i((a, \hat{a}^1, \hat{a}^2), h)) = V^C(x^1, x^2)$. This equation is equivalent to the condition (26). Hence, the proof of condition (27) is provided below. Condition (28) can be proved in the similar way.

Let $\hat{\mathcal{A}} = \{((b_1, q_1), \cdots) \in \mathcal{A}^{C1} \,|\, (\forall t \in \mathcal{T}) \ b_t + \hat{b}^2 \le \frac{2}{r}\}$; obviously $\hat{a}^1 \in \hat{\mathcal{A}}$. $U^1(\mathbf{w}^1((\hat{a}^0, a, \hat{a}^2), h)) = -\infty$ for all $a \in \mathcal{A}^{C1}/\hat{\mathcal{A}}$. (See equations (5) and (6) and footnote 4). Hence, it is sufficient to show that the following condition is satisfied.

$$\hat{a}^1 \in \arg\max_{a \in \hat{\mathcal{A}}} U^1(\mathbf{w}^1((\hat{a}^0, a, \hat{a}^2), \mathbf{x})). \tag{29}$$

In the remainder of this section, $\hat{\mathbf{a}}_a^{-1}$ denotes $(\hat{a}^0, a, \hat{a}^2)$ for arbitrary $a \in \hat{\mathcal{A}}$.

We define the mapping $\hat{T}^\ell$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^C}$ as below.

$$\hat{T}^\ell F(h) = \sup_{(b,\,q) \in \mathcal{X}_h^C} \{\ln(1 - q + z^1) + \ln\left[q + b - (1 + r)x^1\right] + \beta F(b, \hat{b}^2(h))\},$$

where $\mathcal{X}_h^C = \{(b, q) \in \mathbb{R}^2 \mid 1 - q + z^1 \geq 0,\ q + b - (1 + r)x^1 \geq 0,\ (b, \hat{b}^2(\mathbf{x})) \in \mathcal{H}_0\}$.

Furthermore, we define the mapping $\hat{T}^c$: $(\mathbb{R}\cup\{-\infty, \infty\})^{\mathcal{H}_1^C} \to (\mathbb{R}\cup\{-\infty, \infty\})^{\mathcal{H}_0}$ and the operator $\hat{T}$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$.

$$\hat{T}^c F(x^1, x^2) = \sup_{(z^1,z^2) \in \mathcal{S}_1} F(x^1, x^2, z^1, z^2) \quad \text{and} \quad \hat{T}F = \hat{T}^c \hat{T}^\ell F.$$

Lemma B.1 The operator $\hat{T}$ is monotonic, i.e.,

$$(\forall F,\, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}) \quad F \leq G \Rightarrow \hat{T}F \leq \hat{T}G.$$[16]

Proof Suppose $F \leq G$ for $F,\, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$; then the following inequalities hold for arbitrary $(b, q) \in \mathcal{X}_h^C$.

$$\begin{aligned}
&\ln(1 - q + z^1) + \ln\{q + b - (1 + r)x^1\} + \beta F(b, \hat{b}^2(h)) \\
&\leq\ \ln(1 - q + z^1) + \ln\{q + b - (1 + r)x^1\} + \beta G(b, \hat{b}^2(h)) \\
&\leq\ \hat{T}^\ell G(x^1, x^2, z^1, z^2) \leq \hat{T}G(x^1, x^2).
\end{aligned}$$

Hence, $\hat{T}^\ell F(h) \leq \hat{T}G(x^1, x^2)$.
Therefore, $\hat{T}F(x^1, x^2) = \sup_{(z^1,z^2) \in \mathcal{S}_1} \hat{T}^\ell F(x^1, x^2, z^1, z^2) \leq \hat{T}G(x^1, x^2)$. □

We define the functions $\hat{V}^C \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^C}$, W^C, V_0, and $\bar{V}^C \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$ as

$$\hat{V}^C(h) = \sup_{a \in \hat{\mathcal{A}}} U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)), \quad W^C = \hat{T}^c \hat{V}^C, \quad V_0 = \frac{2}{1 - \beta} \ln\left(\frac{2}{r} - x_0^1 - x_0^2\right),$$

$$\bar{V}^C(x^1, x^2) = V_0(x^1, x^2) + \bar{\delta}^C,$$

$$\bar{\delta}^C = \frac{2}{(1 - \beta)^2}\ \{\beta \ln \beta + (1 - \beta) \ln(1 - \beta) + \ln(1 + r) - (1 - \beta) \ln 2\}.$$

Thus, by definition, the following inequality holds.

$$V^C(x^1, x^2) = U^1(\mathbf{w}^1(\hat{\mathbf{a}}, h)) \leq \hat{V}^C(h) \leq W^C(x^1, x^2). \tag{30}$$

[16] For functions f and g in $\mathbb{R}^X$, the inequality $f \leq g$ implies that $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in X$.

Lemma B.2 $(\forall(x^1,x^2)\in\mathcal{H}_0)$ $W^C(x^1,x^2)\le\bar{V}^C(x^1,x^2)$.

Proof For arbitrary $a=((b_1,q_1),(b_2,q_2),\cdots)\in\hat{\mathcal{A}}$ and for arbitrary $h{=}(x^1,x^2,z^1,z^2)\in\mathcal{H}_1^C$, let $\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1},h)=(\{c_t\}_{t\in\mathcal{T}},\{g_t\}_{t\in\mathcal{T}})$. Further, let $\mathbf{x}^i=(x_0^i,x_1^i,\cdots)$, $y^i=(y_1^i,y_2^i,\cdots)$, and $\mathbf{z}=((z_1^1,z_1^2),(z_2^1,z_2^2),\cdots)$ denote the sequence of outstanding local bonds of local government i, that of the local tax of local government i, and that of subsidies determined by $\hat{\mathbf{a}}_a^{-1}$, respectively.

For these sequences, let $\mathbf{x}_{t-1}^i=(x_0^i,\cdots,x_{t-1}^i)$, $\mathbf{y}_{t-1}^i=(y_1^i,\cdots,y_{t-1}^i)$, $\mathbf{z}_t=((z_1^1,z_1^2),\cdots,(z_t^1,z_t^2))$, and $h_t^\ell=((\mathbf{x}_{t-1}^1,\,\mathbf{x}_{t-1}^2),(\mathbf{y}_{t-1}^1,\,\mathbf{y}_{t-1}^2),\,\mathbf{z}_t)$. For arbitrary $t\in\mathcal{T}$,

$$x_t^2=-z_t^2+\frac{1}{2-\beta}\left\{(1+r)[x_{t-1}^2-(1-\beta)x_{t-1}^1]+\frac{2(1-\beta)}{r}-\beta\right\},$$

Therefore, $1+z_t^2+x_t^2-(1+r)x_{t-1}^2=\frac{1-\beta}{2-\beta}I(x_{t-1}^1,x_{t-1}^2)$. Consequently, for arbitrary t, $\{2+x_t^1+x_t^2-(1+r)(x_{t-1}^1+x_{t-1}^2)\}-\{1+z_t^1+x_t^1-(1+r)x_{t-1}^1\}=1+z_t^2+x_t^2-(1+r)x_{t-1}^2\ge 0$. Using the last inequality, for arbitrary t, we have

$$\begin{aligned}\ln c_t^1+\ln g_t^1 &= \ln(1-y_t^1+z_t^1)+\ln\{y_t^1+x_t^1-(1+r)x_{t-1}^1\}\\ &\le 2\ln\{1+z_t^1+x_t^1-(1+r)x_{t-1}^1\}-2\ln 2\\ &\le 2\ln\{2+x_t^1+x_t^2-(1+r)(x_{t-1}^1+x_{t-1}^2)\}-2\ln 2.\end{aligned}$$

The inequality in the second line becomes an equality when $y_t^1=\{1+z_t^1-x_t^1+(1+r)x_{t-1}^1\}/2$. Hence,

$$\begin{aligned}U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1},h)) &= \sum_{t=1}^{\infty}\beta^{t-1}\{\ln c_t^i+\ln g_t^i\}\\ &\le 2\sum_{t=1}^{\infty}\beta^{t-1}\ln\{2+x_t^1+x_t^2-(1+r)(x_{t-1}^1+x_{t-1}^2)\}-\frac{2\ln 2}{1-\beta}.\end{aligned}$$

Applying dynamic programming, for an arbitrary admissible sequence of outstanding of local bonds $((x^1,x_1^1,\cdots),(x^2,x_1^2,\cdots))$,

$$\begin{aligned}&\sum_{t=1}^{\infty}\beta^{t-1}\ln\{2+x_t^1+x_t^2-(1+r)(x_{t-1}^1+x_{t-1}^2)\}\\ &\le \frac{1}{1-\beta}\ln I(x^1,x^2)+\frac{1}{(1-\beta)^2}\{\beta\ln\beta+(1-\beta)\ln(1-\beta)+\beta\ln(1+r)\}\\ &= \frac{1}{2}\bar{V}^C(x^1,x^2)+\frac{\ln 2}{1-\beta}\end{aligned}$$

Accordingly, $U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)) = \bar{V}^C(x^1, x^2)$. Since $a \in \hat{\mathcal{A}}$ is selected arbitrarily, $\hat{V}^C(x^1, x^2, z^1, z^2) \leq \bar{V}^C(x^1, x^2)$ holds for arbitrary $(x^1, x^2, z^1, z^2) \in \mathcal{H}_1^C$. Hence, the lemma follows from inequality (30). □

Lemma B.3 $\lim_{n\to\infty} \hat{T}^n \bar{V}^C(x^1, x^2) = V^C(x^1, x^2)$.

Proof We can verify that for an arbitrary constant number δ, $\hat{T}(V_0 + \delta) = V_0(x^1, x^2) + \delta_0 + \beta\delta$, where $\delta_0 = \frac{2}{1-\beta}\left\{\beta \ln \beta + (1-\beta)\ln(1-\beta) + \ln \frac{1+r}{2-\beta} - (1-\beta)\ln 2\right\}$. Therefore, $\hat{T}^n \bar{V}^C = \hat{T}^n(V_0 + \bar{\delta}) = V_0 + (1 + \beta + \cdots + \beta^{n-1})\delta_0 + \beta^n \bar{\delta}$. Taking n to infinity, we can find $\lim_{n\to\infty} \hat{T}^n \bar{V}^C = V^C$. □

Lemma B.4 $W^C \leq \hat{T} W^C$.

Proof For arbitrary $a = ((b_1, q_1), (b_2, q_2), \cdots) \in \hat{\mathcal{A}}$, let $\hat{U}(h)$ and $\hat{V}^C(x^1, x^2)$ denote $U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h))$ and $\hat{U}(x^1, x^2, \hat{s}^1(x^1, x^2), \hat{s}^2(x^1, x^2))$, respectively. Then, the following inequalities hold.

$$\begin{aligned}
\hat{V}^C(x^1, x^2) &= U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, (x^1, x^2, \hat{s}^1(x^1, x^2), \hat{s}^2(x^1, x^2)))) \\
&\leq \hat{V}^C(x^1, x^2, \hat{s}^1(x^1, x^2), \hat{s}^2(x^1, x^2)) \leq W^C(x^1, x^2). \qquad (31) \\
U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, h)) &= \ln\{1 - q_1(h) + z^1\} + \ln\{q_1(h) + b_1(h) - (1+r)x^1\} \\
&\quad + \beta \hat{U}(b_1(h), b_2(h)) \\
&\leq \hat{T}^\ell \hat{U}(h) \leq \hat{T}\hat{V}^C(x^1, x^2) \leq \hat{T} W^C(x^1, x^2).
\end{aligned}$$

The last inequality follows from Lemma B.1 and inequality (31). Hence,

$$\hat{V}^C(x^1, x^2, z^1, z^2) = \sup_{a \in \hat{\mathcal{A}}} U^1(\mathbf{w}^1(\hat{\mathbf{a}}_a^{-1}, (x^1, x^2, z^1, z^2))) \leq \hat{T} W^C(x^1, x^2).$$

Therefore, $W^C = \sup_{(z^1, z^2) \in \mathcal{S}_1} \hat{V}^C(x^1, x^2, z^1, z^2) \leq \hat{T} W^C$. □

Using Lemmas B.1, B.4, and B.2 repeatedly, we obtain $W^C \leq \hat{T}^n W^C \leq \hat{T}^n \bar{V}^C$. Therefore, $\hat{V}^C(h) \leq W^C \leq \hat{T}^n \bar{V}^C$. The first inequality is obtained from (30). Taking n to infinity and applying Lemma B.3, with inequality (30), we obtain $\hat{V}^C(h) = V^C(x^1, x^2) = U^1(\mathbf{w}^1(\hat{\mathbf{a}}, h))$. This implies condition (29).

C Proof of Proposition 3

Let $\tilde{a}^0, \tilde{a}^1, \tilde{a}^2$ be the strategies constructed from the Markov strategies $(\tilde{s}^1, \tilde{s}^2)$, $(\tilde{b}^1, \tilde{q}^1)$, and $(\tilde{b}^2, \tilde{q}^2)$. Conditions (32), (33), and (34) imply that the combination of strategies $(\tilde{a}^0, \tilde{a}^1, \tilde{a}^2)$ is a Nash equilibrium for an arbitrary subgame because of the recursivity of the model.

$$(\forall h \in \mathcal{H}_1^L) \quad \tilde{a}^0 \in \arg\max_{a \in \mathcal{A}^{D0}} \sum_{i=1,2} U^i(\mathbf{w}^i((a, \tilde{a}^1, \tilde{a}^2), h)). \tag{32}$$

$$(\forall \mathbf{x} \in \mathcal{H}_0) \quad \tilde{a}^1 \in \arg\max_{a \in \mathcal{A}^{D1}} U^1(\mathbf{w}^1((\tilde{a}^0, a, \tilde{a}^2), \mathbf{x})), \tag{33}$$

$$(\forall \mathbf{x} \in \mathcal{H}_0) \quad \tilde{a}^2 \in \arg\max_{a \in \mathcal{A}^{D2}} U^2(\mathbf{w}^2((\tilde{a}^0, \tilde{a}^1, a), \mathbf{x})). \tag{34}$$

Let $\mathbf{x} = (x^1, x^2) \in \mathcal{H}_0$ and $h = ((x_0^1, x_1^1), (x_0^2, x_1^2), y_1^1, y_1^2) \in \mathcal{H}_1^L$; furthermore, let $\tilde{\mathbf{a}}$ denote $(\tilde{a}^0, \tilde{a}^1, \tilde{a}^2)$.

C.1 Proof of Conditions (33) and (34)

The proof of condition (33) is provided below. Condition (34) can be proved in a similar manner. Let $\tilde{\mathcal{A}} = \{((b_1, q_1), \cdots) \in \mathcal{A}^{D1} \,|\, (\forall t \in \mathcal{T})\ b_t + \tilde{b}^2 \leq \frac{2}{r},\ q_t + \tilde{q}^2 \leq 2\}$. By an argument similar to that employed in Section B, it is sufficient to show that the following condition is satisfied.

$$\tilde{a}^1 \in \arg\max_{a \in \tilde{\mathcal{A}}^D} U^1(\mathbf{w}^1((\tilde{a}^0, a, \tilde{a}^2), \mathbf{x})), \tag{35}$$

In the remainder of this section, $\tilde{\mathbf{a}}_a^{-1}$ denotes $(\tilde{a}^0, a, \tilde{a}^2)$ for arbitrary $a \in \tilde{\mathcal{A}}$.

We define the operator[17] T^D: $(\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$ as below.

$$\begin{aligned} & T^D F(x^1, x^2) \\ = & \sup_{(b,q) \in \mathcal{X}_{\mathbf{x}}^D} \left\{ \ln\left[1 - q + \tilde{s}^1(q, \tilde{q}^2(\mathbf{x}))\right] + \ln\left[q + b - (1+r)x^1\right] + \beta F(b, \tilde{b}^2(\mathbf{x})) \right\}, \end{aligned}$$

where $\mathcal{X}_{\mathbf{x}}^D = \{(b, q) \in \mathbb{R}^2 \,|\, q + b - (1+r)x^1 \geq 0,\ (b, \tilde{b}^2(\mathbf{x})) \in \mathcal{H}_0\}$.

Lemma C.1 The operator T^D is monotonic, namely,

$$(\forall F, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}) \quad F \leq G \Rightarrow T^D F \leq T^D G.$$

Proof Suppose $F \leq G$ for $F, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$. The following inequalities hold for arbitrary $(b, q) \in \mathcal{X}_{\mathbf{x}}^D$.

$$\begin{aligned} & \ln\left[1 - q + \tilde{s}^1(q, \tilde{q}^2(x^1, x^2))\right] + \ln\left[q + b - (1+r)x^1\right] + \beta F(b, \tilde{b}^2(x^1, x^2)) \\ \leq \ & \ln\left[1 - q + \tilde{s}^1(q, \tilde{q}^2(x^1, x^2))\right] + \ln\left[q + b - (1+r)x^1\right] + \beta G(b, \tilde{b}^2(x^1, x^2)) \\ \leq \ & T^D G(x^1, x^2). \end{aligned}$$

[17] See Ljungqvist and Sargent (2004, Appendix A).

Therefore, $T^D F(x^1, x^2) \leq T^D G(x^1, x^2)$. □

We define the functions $\tilde{V}^D$, $\bar{V}^D \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$,

$$\tilde{V}^D(x^1, x^2) = \sup_{a \in \tilde{\mathcal{A}}} U^1(\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1}, (x^1, x^2))), \quad \bar{V}^D(x^1, x^2) = V_0(x^1, x^2) + \bar{\delta}^D,$$

$$\bar{\delta}^D = \frac{2}{(1-\beta)^2} \left\{ \beta \ln \beta + (1-\beta) \ln(1-\beta) + \ln(1+r) - \frac{3}{2}(1-\beta) \ln 2 \right\}.$$

Lemma C.2 $V^D(x^1, x^2) = U^1(\mathbf{w}^1(\tilde{\mathbf{a}}, (x^1, x^2))) \leq \tilde{V}^D(x^1, x^2)$.

Proof The Lemma follows from Corollary 2 and the definition of $\tilde{V}^D$. □

Lemma C.3 $\tilde{V}^D(x^1, x^2) \leq \bar{V}^D(x^1, x^2)$.

Proof For arbitrary $a = ((b_1, q_1), (b_2, q_2), \cdots) \in \tilde{\mathcal{A}}$ and for arbitrary $(x^1, x^2) \in \mathcal{H}_0$, let $\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1}, (x^1, x^2)) = (\{c_t\}_{t \in \mathcal{T}}, \{g_t\}_{t \in \mathcal{T}})$. Let $\mathbf{x}^i = (x_0^i, x_1^i, \cdots)$, $\mathbf{y}^i = (y_1^i, y_2^i, \cdots)$, and $\mathbf{z} = ((z_1^1, z_1^2), (z_2^1, z_2^2), \cdots)$ denote the sequence of the outstanding local bonds of local government i, that of the local tax of local government i, and that of the subsidies determined by $\tilde{\mathbf{a}}_a^{-1}$, respectively. From (22) and (23), for arbitrary $t \in \mathcal{T}$,

$$\begin{aligned}
&\{2 + x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2)\} - \{2 - y_t^2 + x_t^1 - (1+r)x_{t-1}^1\} \\
&\quad = y_t^2 + x_t^2 - (1+r)x_{t-1}^2 \\
&\quad = \tilde{q}^2(x_{t-1}^1, x_{t-1}^2) + \tilde{b}^2(x_{t-1}^1, x_{t-1}^2) - (1+r)x_{t-1}^2 \\
&\quad = \frac{1-\beta}{3-\beta} I(x_{t-1}^1, x_{t-1}^2) \geq 0,
\end{aligned}$$

Since $z_t^1 = \tilde{s}^1(y_t^1, y_t^2) = \frac{y_t^1 - y_t^2}{2}$, using the last inequality, for arbitrary t, we have

$$\begin{aligned}
\ln c_t + \ln g_t &= \ln(1 - y_t^1 + z_t^1) + \ln\{y_t^1 + x_t^1 - (1+r)x_{t-1}^1\} \\
&= \ln(1 - y_t^1 + \frac{y_t^1 - y_t^2}{2}) + \ln\{y_t^1 + x_t^1 - (1+r)x_{t-1}^1\} \\
&\leq 2 \ln\{2 - y_t^2 + x_t^1 - (1+r)x_{t-1}^1\} - 3 \ln 2 \\
&\leq 2 \ln\{2 + x_t^1 + x_t^2 - (1+r)(x_{t-1}^1 + x_{t-1}^2)\} - 3 \ln 2.
\end{aligned}$$

The inequality in the third line becomes an equality when $y_t^1 = 1 - \frac{1}{2} y_t^2 - \frac{1}{2}\{x_t^1 -$

$(1+r)x^2_{t-1}\}$. Hence,

$$
\begin{aligned}
U^1(\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1},(x^1,x^2))) &= \sum_{t=1}^{\infty}\beta^{t-1}\{\ln c_t^i + \ln g_t^i\} \\
&\leq 2\sum_{t=1}^{\infty}\beta^{t-1}\ln\{2+x_t^1+x_t^2-(1+r)(x_{t-1}^1+x_{t-1}^2)\} - \frac{3\ln 2}{1-\beta}.
\end{aligned}
$$

Applying dynamic programming, for an arbitrary admissible sequence of outstanding of local bonds $((x^1,x_1^1,\cdots),(x^2,x_1^2,\cdots))$,

$$
\begin{aligned}
&\sum_{t=1}^{\infty}\beta^{t-1}\ln\{2+x_t^1+x_t^2-(1+r)(x_{t-1}^1+x_{t-1}^2)\} \\
&\leq \frac{1}{1-\beta}\ln I(x^1,x^2) + \frac{1}{(1-\beta)^2}\{\beta\ln\beta+(1-\beta)\ln(1-\beta)+\beta\ln(1+r)\} \\
&= \frac{1}{2}V^D(x^1,x^2) + \frac{3\ln 2}{2(1-\beta)}.
\end{aligned}
$$

Accordingly, $U^1(\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1},(x^1,x^2))) \leq \bar{V}^D(x^1,x^2)$. Since $a\in\tilde{\mathcal{A}}$ is selected arbitrarily, the lemma follows. □

Lemma C.4 $\lim_{n\to\infty}(T^D)^n\bar{V}^D(x^1,x^2) = V^D(x^1,x^2)$.

Proof We can verify that for an arbitrary constant number δ, $T^D(V_0+\delta) = V_0(x^1,x^2)+\delta_0^D+\beta\delta$, where $\delta_0^D = \frac{2}{1-\beta}\left\{\beta\ln\beta+(1-\beta)\ln(1-\beta)+\ln\frac{1+r}{3-\beta}+\frac{3\beta-1}{2}\ln 2\right\}$.

Therefore, $(T^D)^n\bar{V}^D = (T^D)^n(V_0+\bar{\delta}^D) = V_0+(1+\beta+\cdots+\beta^{n-1})\delta_0^D+\beta^n\bar{\delta}^D$. Taking n to infinity, we can find $\lim_{n\to\infty}(T^D)^n\bar{V}^D = V^D$. □

Lemma C.5 $\tilde{V}^D \leq T^D\tilde{V}^D$.

Proof For arbitrary $a=((b_1,q_1),(b_2,q_2),\cdots)\in\tilde{\mathcal{A}}$, $V_a^D(\mathbf{x})$ denotes $U^1(\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1},\mathbf{x}))$.

$$
\begin{aligned}
U^1(\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1},\mathbf{x})) &= \ln\{1-q_1(\mathbf{x})+\tilde{s}^1(q_1(\mathbf{x}),\tilde{q}^2(\mathbf{x}))\} \\
&\quad +\ln\{q_1(\mathbf{x})+b_1(\mathbf{x})-(1+r)x^1\}+\beta V_a^D(b_1(\mathbf{x}),\tilde{b}^2(\mathbf{x})) \\
&\leq T^DV_a^D(\mathbf{x}) \leq T^D\tilde{V}^D(\mathbf{x}).
\end{aligned}
$$

The last inequality follows from Lemma C.1 and the definition of $\tilde{V}^D$. Since $a\in\tilde{\mathcal{A}}$ is selected arbitrarily, $\tilde{V}^D(\mathbf{x}) = \sup_{a\in\tilde{\mathcal{A}}}U^1(\mathbf{w}^1(\tilde{\mathbf{a}}_a^{-1},\mathbf{x})) \leq T^D\tilde{V}^D(\mathbf{x})$. □

Using Lemmas C.1, C.5, and C.3 repeatedly, we obtain $\tilde{V}^D \leq (T^D)^n\tilde{V}^D \leq (T^D)^n\bar{V}^D$. Taking n to infinity and using Lemma C.4, with Lemma C.2, we obtain $\bar{U}(x^1,x^2) = V^D(x^1,x^2) = U^1(\mathbf{w}^1(\tilde{\mathbf{a}},(x^1,x^2)))$. This implies condition (35).

C.2 Proof of Condition (32)

For arbitrary $a \in \mathcal{A}^{D0}$, $(a, \tilde{a}^1, \tilde{a}^2)$ is denoted by $\tilde{\mathbf{a}}_a^{-0}$. We define the mappings $\tilde{T}^\ell$ as follows: $(\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0}$, $\tilde{T}^c$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_0} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L}$, and define the operator $\tilde{T}$: $(\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L} \to (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L}$ as below.

$$\begin{aligned}
\tilde{T}^\ell F(x_0^1, x_0^2) &= F(x_0^1, \tilde{b}^1(x_0^1, x_0^2), x_0^2, \tilde{b}^2(x_0^1, x_0^2), \tilde{q}^1(x_0^1, x_0^2), \tilde{q}^2(x_0^1, x_0^2)), \\
\tilde{T}^c G(h) &= 2\ln\left(1 - \frac{y_1^1 + y_1^2}{2}\right) + \ln\{y_1^1 + x_1^1 - (1+r)x_0^1\} \\
&\quad + \ln[y_1^2 + x_1^2 - (1+r)x_0^2] + \beta G(x_1^1, x_1^2), \\
\tilde{T}F &= \tilde{T}^c \tilde{T}^\ell F.
\end{aligned}$$

Lemma C.6 The operator $\tilde{T}$ is monotone, namely,

$$(\forall F,\, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L}) \quad F \leq G \Rightarrow \tilde{T}F \leq \tilde{T}G.$$

Proof Suppose $F \leq G$ for $F, G \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L}$. Since $\tilde{T}^\ell F \leq \tilde{T}^\ell G$, the following inequalities hold for arbitrary $h \in \mathcal{H}_1^L$ and arbitrary $(z^1, z^2) \in \mathcal{S}_1$.

$$\begin{aligned}
\tilde{T}F(h) &= 2\ln\left(1 - \frac{y_1^1 + y_1^2}{2}\right) + \ln\{y_1^1 + x_1^1 - (1+r)x_0^1\} \\
&\quad + \ln[y_1^2 + x_1^2 - (1+r)x_0^2] + \beta \tilde{T}^\ell F(x_1^1, x_1^2) \\
&\leq 2\ln\left(1 - \frac{y_1^1 + y_1^2}{2}\right) + \ln\{y_1^1 + x_1^1 - (1+r)x_0^1\} \\
&\quad + \ln[y_1^2 + x_1^2 - (1+r)x_0^2] + \beta \tilde{T}^\ell G(x_1^1, x_1^2).
\end{aligned}$$

The last equation is equal to $\tilde{T}G(h)$. □

We define the functions W^D, $\tilde{W}^D$, $\bar{W}^D \in (\mathbb{R} \cup \{-\infty, \infty\})^{\mathcal{H}_1^L}$,

$$W^D(h) = \sup_{a \in \mathcal{A}^{D0}} \sum_{i=1}^{2} U^i(\mathbf{w}^i(\tilde{\mathbf{a}}_a^{-0}, h)),$$

$$\tilde{W}^D(h) = 2\ln\{1 - \frac{y_1^1 + y_1^2}{2}\} + \sum_{i=1}^{2} \ln\{y_1^i + x_1^i - (1+r)x_0^i\} + 2\beta V^D(x_1^1, x_1^2),$$

$$\bar{W}^D(h) = 2\ln\{1 - \frac{y_1^1 + y_1^2}{2}\} + \sum_{i=1}^{2} \ln\{y_1^i + x_1^i - (1+r)x_0^i\} + 2\beta V^*(x_1^1, x_1^2).$$

Lemma C.7 $\tilde{W}^D(h) = U^1(\mathbf{w}^1(\tilde{\mathbf{a}}, h)) + U^2(\mathbf{w}^2(\tilde{\mathbf{a}}, h)) \leq W^D(h)$.

Proof From Lemma C.2, $U^1(\mathbf{w}^1(\tilde{\mathbf{a}}, (x_1^1, x_1^2))) = V^D(x_1^1, x_1^2)$. Furthermore, it can be verified that $U^2(\mathbf{w}^2(\tilde{\mathbf{a}}, (x_1^1, x_1^2))) = V^D(x_1^1, x_1^2)$ by a proof similar to that of Lemma C.2. Hence, for $i = 1, 2$,

$$\begin{aligned} U^i(\mathbf{w}^i(\tilde{\mathbf{a}}, h)) &= \ln(1 - y_1^i + \frac{y_1^i - y_1^j}{2}) + \ln\{y_1^i + x_1^i - (1+r)x_0^i\} \\ &\qquad +\beta U^i(\mathbf{w}^i(\tilde{\mathbf{a}}, (x_1^1, x_1^2))) \\ &= \ln(1 - \frac{y_1^1 + y_1^2}{2}) + \ln\{y_1^i + x_1^i - (1+r)x_0^i\} + \beta V^D(x_1^1, x_1^2). \end{aligned}$$

The equality in the Lemma follows, and the inequality follows by the definition. □

Lemma C.8 $W^D(h) \leq \bar{W}^D(h)$.

Proof For all $(x_1^1, x_1^2) \in \mathcal{H}_0$ and for all $h_1 = ((x_0^1, x^1), (x_0^2, x^2), (y_1^1, y_1^2), (z_1^1, z_1^2)) \in \mathcal{H}_1$, such that $(x^1, x^2) = (x_1^1, x_1^2)$, $2V^*(x_1^1, x_1^2) \geq \sum_{i=1}^2 U^i(\mathbf{w}^i(\mathbf{a}, h_1))$ for arbitrary $\mathbf{a} \in \mathcal{A}^D$, since V^* is the value function corresponding to the optimal solutions, and $\mathbf{w}^i(\mathbf{a}, h_1)$ represents the sequence of consumption after period two.

Furthermore, for $(y_1^1, y_1^2) \in \mathcal{Q}_1$,

$$\max_{(z^1,z^2)\in\mathcal{S}_1} \{\ln(1 - y_1^1 + z^1) + \ln(1 - y_1^2 + z^2)\} = 2\ln(1 - \frac{y_1^1 + y_1^2}{2}). \tag{36}$$

Hence, for arbitrary $a = ((s_1^1, s_1^2), (s_2^1, s_2^2), \cdots) \in \mathcal{A}^{D0}$,

$$\begin{aligned} \sum_{i=1}^2 U^i(\mathbf{w}^i(\tilde{\mathbf{a}}_a^{-0}, h)) &= \sum_{i=1}^2 \{\ln[1 - y_1^i + s_1^i(h)] + \ln[y_1^i + x_1^i - (1+r)x_0^i] \\ &\qquad +\beta U^i(\mathbf{w}^i(\tilde{\mathbf{a}}_a^{-0}, h_1))\} \\ &\leq 2\ln(1 - \frac{y_1^1 + y_1^2}{2}) + \sum_{i=1}^2 \ln\{y_1^i + x_1^i - (1+r)x_0^i\} + 2\beta V^*(x_1^1, x_1^2) = \bar{W}^D(h), \end{aligned}$$

where, in the first equation, $h_1 = ((x_0^1, x_1^1), (x_0^2, x_1^2), (y_1^1, y_1^2), (s_1^1(h), s_1^2(h)))$.

Since $a \in \mathcal{A}^{D0}$ is selected arbitrarily, the lemma follows. □

Lemma C.9 $\lim_{n\to\infty} \tilde{T}^n \bar{W}^D = \tilde{W}^D$.

Proof We can verify that for an arbitrary constant number δ, $\tilde{T}^\ell \tilde{T}^c(2V_0 + 2\delta) = 2V_0 + 2\delta_0^D + 2\beta\delta$. Therefore,

$$\begin{aligned} (\tilde{T}^\ell \tilde{T}^c)^n(2V^*) &= (\tilde{T}^\ell \tilde{T}^c)^n(2V_0 + 2\delta^*) = 2V_0 + 2(1 + \beta + \cdots + \beta^{n-1})\delta_0^D + 2\beta^n\delta^* \\ &= 2V_0 + 2(1 - \beta^n)\delta^D + 2\beta^n\delta^*. \end{aligned}$$

Note that $\bar{W}^D = \tilde{T}^c(2V^*)$,

$$\begin{aligned}
\tilde{T}^n\bar{W}^D &= \tilde{T}^n\tilde{T}^c(2V^*) = (\tilde{T}^c\tilde{T}^\ell)^n\tilde{T}^c(2V^*) = \tilde{T}^c(\tilde{T}^\ell\tilde{T}^c)^n(2V^*) \\
&= \tilde{T}^c(2V_0 + 2(1-\beta^n)\delta^D + 2\beta^n\delta^*) \\
&= 2\ln\left(1 - \frac{y_1^1 + y_1^2}{2}\right) + \ln\{y_1^1 + x_1^1 - (1+r)x_0^1\} \\
&\qquad + \ln[y_1^2 + x_1^2 - (1+r)x_0^2] + 2\beta\{V_0 + (1-\beta^n)\delta^D + \beta^n\delta^*\}.
\end{aligned}$$

Taking n to infinity, we can find

$$\begin{aligned}
\lim_{n\to\infty}\tilde{T}^n\bar{W}^D &= 2\ln\left(1 - \frac{y_1^1 + y_1^2}{2}\right) + \ln\{y_1^1 + x_1^1 - (1+r)x_0^1\} \\
&\qquad + \ln[y_1^2 + x_1^2 - (1+r)x_0^2] + 2\beta(V_0 + \delta^D) \\
&= 2\ln\left(1 - \frac{y_1^1 + y_1^2}{2}\right) + \ln\{y_1^1 + x_1^1 - (1+r)x_0^1\} \\
&\qquad + \ln[y_1^2 + x_1^2 - (1+r)x_0^2] + 2\beta\bar{V}^D.
\end{aligned}$$

The last equation in the inequalities is equal to $\bar{W}^D$. □

Given $a = ((s_1^1, s_1^2), \cdots) \in \tilde{\mathcal{A}}$ and $\bar{h} = ((\bar{x}_0^1, \bar{x}_1^1), (\bar{x}_0^2, \bar{x}_1^2), \bar{y}_1^1, \bar{y}_1^2) \in \mathcal{H}_1^L$, we define $\bar{h}_1 \in \mathcal{H}_1$, $h' \in \mathcal{H}_1^L$, and $\bar{a} = ((\bar{s}_1^1, \bar{s}_1^2), \cdots) \in \mathcal{A}^{D0}$ as follows.

$$\bar{h}_1 = ((\bar{x}_0^1, \bar{x}_1^1), (\bar{x}_0^2, \bar{x}_1^2), \bar{y}_1^1, \bar{y}_1^2, (s_1^1(\bar{h}), s_1^2(\bar{h}))),$$

$$h' = ((\bar{x}_1^1, \tilde{b}^1(\bar{x}_1^1, \bar{x}_1^2)), (\bar{x}_1^2, \tilde{b}^2(\bar{x}_1^1, \bar{x}_1^2)), \tilde{q}^1(\bar{x}_1^1, \bar{x}_1^2), \tilde{q}^1(\bar{x}_1^1, \bar{x}_1^2)),$$

$$(\forall t \in \mathcal{T})\,(\forall h_t \in \mathcal{H}_t^L)\;\; \bar{s}_t^i(h_t) = s_{t+1}^i(\bar{h}_{t+1}),$$

where, for $h_1 = ((x_0^1, x_1^1), (x_0^2, x_1^2), y^1, y^2)$,

$$\bar{h}_2 = ((\bar{x}_0^1, \bar{x}_1^1, x_1^1), (\bar{x}_0^2, \bar{x}_1^2, x_1^2), (\bar{y}_1^1, y^1), (\bar{y}_1^2, y^2), (s_1^1(\bar{h}), s_1^2(\bar{h}))),$$

and, for $h_t = (\mathbf{x}_t^1, \mathbf{x}_t^2, \mathbf{y}_t^1, \mathbf{y}_t^2, \mathbf{z}_{t-1})(t \in \mathcal{T}/\{1\})$ and $\mathbf{x}_t^i = (x_0^i, x_1^i, \cdots, x_t^i)(i = 1, 2)$,[18]

$$\bar{h}_{t+1} = (\bar{\mathbf{x}}_{t+1}^1, \bar{\mathbf{x}}_{t+1}^2, \bar{\mathbf{y}}_{t+1}^1, \bar{\mathbf{y}}_{t+1}^2, \bar{\mathbf{z}}_t),\;\; \bar{\mathbf{x}}_{t+1}^i = (\bar{x}_0^i, \bar{x}_1^i, x_1^i, \cdots, x_t^i)\;\;(i = 1, 2),$$

$$\bar{\mathbf{y}}_{t+1}^i = (\bar{y}_1^i, \mathbf{y}_t^i)\;\;(i = 1, 2),\;\; \bar{\mathbf{z}}_t = ((s_1^1(\bar{h}), s_1^2(\bar{h})), \mathbf{z}_{t-1}).$$

Lemma C.10 The following equation holds.

$$(\forall a \in \tilde{\mathcal{A}})(\forall \bar{h} \in \mathcal{H}_1^L)(\forall i = 1, 2)\;\; \mathbf{w}^i((a, \tilde{a}^1, \tilde{a}^2), \bar{h}_1) = \mathbf{w}^i((\bar{a}, \tilde{a}^1, \tilde{a}^2), h'), \qquad (37)$$

[18] $\bar{\mathbf{x}}_{t+1}^i = (\bar{x}_0^i, \bar{x}_1^i, \frac{1}{r}, \cdots, \frac{1}{r})$ if $\bar{x}_0^1 + \bar{x}_0^2 = \frac{2}{r}$ or $\bar{x}_1^1 + \bar{x}_1^2 = \frac{2}{r}$.

Proof Let $(\{\bar{c}_t^i\}_{t\in\mathcal{T}}, \{\bar{g}_t^i\}_{t\in\mathcal{T}})$ denote $\mathbf{w}^i(\tilde{\mathbf{a}}_a^{-0}, \bar{h})$. For all $t \in \mathcal{T}$, $\bar{c}_t^i$ and $\bar{g}_t^i$ are determined by $\bar{c}_t^i = 1 - \bar{y}_t^i + \bar{z}_t^i$, $\bar{g}_t^i = \bar{y}_t^i + \bar{x}_t^i - (1+r)\bar{x}_{t-1}^i$, where $\bar{x}_t^i$, $\bar{y}_t^i$, and $\bar{z}_t^i$ are given in the following way.

$$\bar{x}_t^i = \tilde{b}^i(\bar{x}_{t-1}^1, \bar{x}_{t-1}^2),\ \bar{y}_t^i = \tilde{q}^i(\bar{x}_{t-1}^1, \bar{x}_{t-1}^2),\ \bar{\mathbf{x}}_1^i = (\bar{x}_0^i, \bar{x}_1^i),\ \bar{\mathbf{y}}_1^i = \bar{y}_1^i,$$

$$\bar{\mathbf{x}}_{t+1}^i = (\bar{\mathbf{x}}_t^i, \bar{x}_{t+1}^i),\ \bar{\mathbf{y}}_{t+1}^i = (\bar{\mathbf{y}}_t^i, \bar{y}_{t+1}^i),\ \bar{z}_1^i = s_1^i(\bar{h}),\ \bar{\mathbf{z}}_1 = (\bar{z}_1^1, \bar{z}_1^2),$$

$$\bar{z}_{t+1}^i = s_{t+1}^i(\bar{\mathbf{x}}_{t+1}^1, \bar{\mathbf{x}}_{t+1}^2, \bar{\mathbf{y}}_{t+1}^1, \bar{\mathbf{y}}_{t+1}^2, \bar{\mathbf{z}}_t),\ \bar{\mathbf{z}}_{t+1} = (\bar{\mathbf{z}}_t, (\bar{z}_{t+1}^1, \bar{z}_{t+1}^2)).$$

The left hand side of equation (37) represents the sequence of consumption of the original sequence from period two, that is, $\mathbf{w}^i((a, \tilde{a}^1, \tilde{a}^2), \bar{h}_1) = (\{\bar{c}_{t+1}^i\}_{t\in\mathcal{T}}, \{\bar{g}_{t+1}^i\}_{t\in\mathcal{T}})$.

Let $\mathbf{w}^i((\bar{a}, \tilde{a}^1, \tilde{a}^2), h') = (\{c_t^{i\prime}\}_{t\in\mathcal{T}}, \{g_t^{i\prime}\}_{t\in\mathcal{T}})$. For $t \in \mathcal{T}$, $c_t^{i\prime}$ and $g_t^{i\prime}$ are determined as follows: $c_t^{i\prime} = 1 - y_t^{i\prime} + z_t^{i\prime}$ and $g_t^{i\prime} = y_t^{i\prime} + x_t^{i\prime} - (1+r)x_{t-1}^{i\prime}$ for arbitrary $t \in \mathcal{T}$ where $x_0^{i\prime} = \bar{x}_1^i$, $z_1^{i\prime} = \bar{s}_1^i(h')$, and $x_t^{i\prime}$, $y_t^{i\prime}$, and $z_t^{i\prime}$ are given in the following way.

$$x_t^{i\prime} = \tilde{b}^i(x_{t-1}^{1'}, x_{t-1}^{2'}), \quad y_t^{i\prime} = \tilde{q}^i(x_{t-1}^{1'}, x_{t-1}^{2'}), \tag{38}$$

$$\mathbf{x}_t^{i'} = (\mathbf{x}_{t-1}^{i'}, x_t^{i'}), \quad \mathbf{y}_{t+1}^{i'} = (\mathbf{y}_t^{i'}, y_{t+1}^{i'}),$$

$$z_{t+1}^{i'} = \bar{s}_{t+1}^i(\mathbf{x}_{t+1}^{1'}, \mathbf{x}_{t+1}^{2'}, \mathbf{y}_{t+1}^{1'}, \mathbf{y}_{t+1}^{2'}, \mathbf{z}_t'), \quad \mathbf{z}_{t+1}' = (\mathbf{z}_t', (z_{t+1}^{1'}, z_{t+1}^{2'})),$$

where $\mathbf{x}_0^{i\prime} = \bar{x}_1^1$, $\mathbf{y}_1^{i\prime} = \bar{y}_1^1$, and $\mathbf{z}_1' = (z_1^{1\prime}, z_1^{2\prime})$. From (38) and $x_0^{i\prime} = \bar{x}_1^i$, $x_1^{i\prime} = \tilde{b}^i(x_0^{1'}, x_0^{2'}) = \tilde{b}^i(\bar{x}_1^1, \bar{x}_1^2) = \bar{x}_2^i$ for $i = 1, 2$. Hence, $x_2^{i\prime} = \tilde{b}^i(x_1^{1'}, x_1^{2'}) = \tilde{b}^i(\bar{x}_2^1, \bar{x}_2^2) = \bar{x}_3^i$. Similarly, it follows that $x_t^{i\prime} = \bar{x}_{t+1}^i$, $y_t^{i\prime} = \bar{y}_{t+1}^i$ for arbitrary $t \in \mathcal{T}$ and $i = 1, 2$. From the definition of $\bar{a}$, $z_1^{i\prime} = \bar{s}_1^i(h') = s_2^i(\bar{h}_2')$ where

$$\begin{aligned}\bar{h}_2' &= ((\bar{x}_0^1, \bar{x}_1^1, \tilde{b}^1(\bar{x}_1^1, \bar{x}_1^2)), (\bar{x}_0^2, \bar{x}_1^2, \tilde{b}^2(\bar{x}_1^1, \bar{x}_1^2)), (\bar{y}_1^1, \tilde{q}^1(\bar{x}_1^1, \bar{x}_1^2)), \\ &\qquad (\bar{y}_1^2, \tilde{q}^2(\bar{x}_1^1, \bar{x}_1^2)), (s_1^1(\bar{h}), s_1^2(\bar{h}))) \\ &= ((\bar{x}_0^1, \bar{x}_1^1, \bar{x}_2^1), (\bar{x}_0^2, \bar{x}_1^2, \bar{x}_2^2), (\bar{y}_1^1, \bar{y}_2^1), (\bar{y}_1^2, \bar{y}_2^2), (\bar{z}_1^1, \bar{z}_1^2)).\end{aligned}$$

Therefore, $z_1^{i\prime} = \bar{z}_2^i$. By a similar argument, it can be verified that $z_t^{i\prime} = \bar{z}_{t+1}^i$ for all $t \in \mathcal{T}$ and for $i = 1, 2$. Hence, for all $t \in \mathcal{T}$ and for $i = 1, 2$,

$$c_t^{i\prime} = 1 - y_t^{i\prime} + z_t^{i\prime} = 1 - \bar{y}_{t+1}^i + \bar{z}_{t+1}^i = \bar{c}_{t+1}^i,$$

$$g_t^{i\prime} = y_t^{i\prime} + x_t^{i\prime} - (1+r)x_{t-1}^{i\prime} = \bar{y}_{t+1}^i + \bar{x}_{t+1}^i - (1+r)\bar{x}_t^i = \bar{g}_{t+1}^i.$$

Hence, (37) follows. □

Lemma C.11 $W^D \le \tilde{T} W^D$.

Proof For arbitrary $a = ((s_1^1, s_1^2), \cdots) \in \mathcal{A}^{D0}$ and $\bar{h} = ((\bar{x}_0^1, \bar{x}_1^1), (\bar{x}_0^2, \bar{x}_1^2), \bar{y}_1^1, \bar{y}_1^2) \in \mathcal{H}_1^L$,

$$
\begin{aligned}
&\sum_{i=1}^{2} U^i(\mathbf{w}^i(\tilde{\mathbf{a}}_a^{-0}, \bar{h})) \\
&= \sum_{i=1}^{2} \{\ln(1 - \bar{y}_1^i + s_1^i(\bar{h})) + \ln(\bar{y}_1^i + \bar{x}_1^i - (1+r)\bar{x}_0^1)\} + \beta \sum_{i=1}^{2} U^i(\mathbf{w}^i(\tilde{\mathbf{a}}_a^{-0}, \bar{h}_1))) \\
&\le 2\ln\left(1 - \frac{\bar{y}_1^1 + \bar{y}_1^2}{2}\right) + \sum_{i=1}^{2} \{\ln(\bar{y}_1^i + \bar{x}_1^i - (1+r)\bar{x}_0^1)\} + \beta \sum_{i=1}^{2} U^i(\mathbf{w}^i(\tilde{\mathbf{a}}_{\bar{a}}^{-0}, h'))) \\
&\le 2\ln\left(1 - \frac{\bar{y}_1^1 + \bar{y}_1^2}{2}\right) + \sum_{i=1}^{2} \{\ln(\bar{y}_1^i + \bar{x}_1^i - (1+r)\bar{x}_0^1)\} + \beta W^D(h') \\
&= 2\ln\left(1 - \frac{\bar{y}_1^1 + \bar{y}_1^2}{2}\right) + \sum_{i=1}^{2} \{\ln(\bar{y}_1^i + \bar{x}_1^i - (1+r)\bar{x}_0^1)\} + \beta \tilde{T}^\ell W^D(\bar{x}_1^1, \bar{x}_1^2) \\
&= \tilde{T} W^D(\bar{h}).
\end{aligned}
$$

The inequality in the second line follows from (36) and Lemma C.10. Since $a \in \mathcal{A}^{D0}$ is selected arbitrarily, the lemma follows. □

Using Lemma C.6, Lemma C.8, and Lemma C.11 repeatedly, we obtain the inequalities $W^D \le \tilde{T}^n W^D \le \tilde{T}^n \bar{W}^D$. Taking n to infinity and using Lemma C.9, with Lemma C.7, we obtain $W^D(h) = \tilde{W}^D(h) = U^1(\mathbf{w}^1(\tilde{\mathbf{a}}, h)) + U^2(\mathbf{w}^2(\tilde{\mathbf{a}}, h))$. This implies condition (32).

D Equilibrium of T-period Models

T-period CL Model The subgame perfect equilibrium of the T-period CL model under which the payoffs, namely, the utility of all the residents, are finite satisfies the following equations.[19]

$$
s_T^i = \frac{1}{2}\left\{(\bar{x}^j - (1+r)x_{T-1}^j) - (\bar{x}^i - (1+r)x_{T-1}^i)\right\} (j \ne i),
$$

$$
q_t^i = \begin{cases} (1 + z_t^i) - \frac{1}{2}\lambda_t \cdot I_t(x_{t-1}^1, x_{t-1}^2) & \text{if } t = 1, ..., T-1 \\ \frac{1}{2}\left\{(1 + z_T^i) - [\bar{x}^i - (1+r)x_{T-1}^i]\right\} & \text{if } t = T, \end{cases}
$$

[19]The notations are the same as in Section 2.

$$b_t^i = \begin{cases} -(1+z_t^i) + (1+r)x_{t-1}^i + \lambda_t \cdot I_t(x_{t-1}^1, x_{t-1}^2) & \text{if } t = 1, ..., T-1 \\ \bar{x}^i & \text{if } t = T, \end{cases}$$

where $\lambda_t = \left(2 + \sum_{\tau=1}^{T-t} \beta^\tau\right)^{-1}$. Conversely, a combination of strategies satisfying these conditions is a subgame perfect equilibrium. Although there is a continuum of subgame perfect equilibria, private and local public goods consumption paths are the same and given as follows:

$$c_t^1 = c_t^2 = g_t^1 = g_t^2 = \frac{1}{2}\lambda_t \cdot I_t(x_{t-1}^1, x_{t-1}^2),\ I_{t+1}(x_t^1, x_t^2) = (1+r)(1-\lambda_t) \cdot I_t(x_{t-1}^1, x_{t-1}^2),$$

where $\lambda_T \equiv 2^{-1}$. Similarly, the value function is the same for all the subgame perfect equilibria and given as follows:

$$V_T(x_0^1, x_0^2) = \frac{2(1-\beta^T)}{1-\beta} \ln I_1(x_0^1, x_0^2) + C_T,$$

$$C_T = 2\sum_{t=1}^{T} \beta^{t-1} \ln \lambda_t (1+r)^{t-1} \prod_{\tau=1}^{t-1} (1-2\lambda_\tau) - \frac{2(1-\beta^T)}{1-\beta} \ln 2,$$

where $\prod_{\tau=1}^{0}(1-2\lambda_\tau) = 1$. It can be easily verified that $(\forall t \in \mathcal{T})\ \lim_{T\to\infty} \lambda_t = \frac{1-\beta}{2-\beta}$ and that $(\forall t \in \mathcal{T})(\forall (x^1, x^2) \in \mathcal{H}_0)\ \ \lim_{T\to\infty} V_T(x^1, x^2) = V(x^1, x^2)$. Hence, for CL models, Proposition 4 follows.

***T*-period DL Model** The subgame perfect equilibrium of the T-period DL model under which the payoffs profile is finite-valued satisfies the following equations.[20]

$$(\forall i = 1,2)\,(\forall t = 1, ..., T)\ \ s_t^i = \frac{1}{2}(y_t^i - y_t^j)\ \ (j \neq i),$$

$$(\forall i = 1,2)\,(\forall t = 1, ..., T)\ \ q_t^i = \mu_t \cdot I_t(x_{t-1}^1, x_{t-1}^2) + (1+r)x_{t-1}^i - b_t^i,$$

$$(\forall t = 1, ..., T)\ \ b_t^1 + b_t^2 = \frac{2}{r} + 3\mu_t \cdot I_t(x_{t-1}^1, x_{t-1}^2) - I(x_{t-1}^1, x_{t-1}^2),$$

$$(\forall i = 1,2)\ \ b_T^i = \bar{x}^i,$$

where $\mu_T \equiv 3^{-1}$ and for $t = 1, ..., T-1$, $\mu_t = \left(3 + 2\sum_{\tau=1}^{T-t} \beta^\tau\right)^{-1}$. Conversely, a combination of strategies satisfying these conditions is a subgame perfect equilibrium. Although there is a continuum of subgame perfect equilibria, private and local public goods consumption paths are the same and given as follows:

$$2c_t^1 = 2c_t^2 = g_t^1 = g_t^2 = \mu_t I_t(x_{t-1}^1, x_{t-1}^2),\ I_{t+1}(x_t^1, x_t^2) = (1+r)(1-3\mu_t) \cdot I_t(x_{t-1}^1, x_{t-1}^2).$$

[20] The notations are the same as in Section 3.

Similarly, the value function is the same for all the subgame perfect equilibria and is given as follows:

$$V_T^D(x_0^1, x_0^2) = \frac{2(1-\beta^T)}{1-\beta} \ln I_1(x_0^1, x_0^2) + C_T^D,$$

$$C_T^D = 2\sum_{t=1}^{T} \beta^{t-1} \ln \mu_t (1+r)^{t-1} \prod_{\tau=1}^{t-1} (1-3\mu_\tau) - \frac{1-\beta^T}{1-\beta} \ln 2,$$

where $\prod_{\tau=1}^{0}(1-3\mu_\tau) = 1$. It can be easily verified that $(\forall t \in \mathcal{T})$ $\lim_{T\to\infty} \mu_t = \frac{1-\beta}{3-\beta}$ and that $(\forall t \in \mathcal{T})(\forall (x^1, x^2) \in \mathcal{H}_0)$ $\lim_{T\to\infty} V_T^D(x^1, x^2) = V^D(x^1, x^2)$. Hence, Proposition 4 follows.

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